

DERIVATIONS OF LIE ALGEBRAS OF DOMINANT UPPER TRIANGULAR LADDER MATRICES

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ABSTRACT. We explicitly describe the Lie algebras $M_{\mathcal{L}}$ of ladder matrices in M_n associate with dominant upper triangular ladders \mathcal{L} , and completely characterize the derivations of these $M_{\mathcal{L}}$ over a field \mathbb{F} with $\text{char}(\mathbb{F}) \neq 2$. We also completely characterize the derivations of Lie algebras $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ where \mathcal{L} are strongly dominant upper triangular ladders and $\text{char}(\mathbb{F}) \neq 2, 3$.

Keywords: ladder matrix; upper triangular ladder; derivation algebra

1. INTRODUCTION

Ladder matrix is a natural extension of block upper triangular matrix. A ladder matrix is one that has zero entries outside of a ladder shape region. Let $[n] := \{1, 2, \dots, n\}$. Given a field \mathbb{F} , let M_{mn} be the set of $m \times n$ matrices over \mathbb{F} , and $M_n := M_{nn}$. We define a partial order on $\mathbb{Z}^+ \times \mathbb{Z}^+$: (i_1, j_1) is said to *dominate* (i_2, j_2) , written as $(i_1, j_1) \succeq (i_2, j_2)$, whenever $i_1 \geq i_2$ and $j_1 \leq j_2$.

Definition 1.1. A subset $\mathcal{L} := \{(i_1, j_1), \dots, (i_s, j_s)\}$ of the index set $[n] \times [n]$ of M_n is called a ladder of step s and size n , if

$$i_1 < i_2 < \dots < i_s \quad \text{and} \quad j_1 < j_2 < \dots < j_s.$$

Each (i_ℓ, j_ℓ) ($\ell \in [s]$) is called a corner point of \mathcal{L} . The set $M_{\mathcal{L}}$ of \mathcal{L} -ladder matrices is a subset of M_n defined by $M_{\emptyset} = \{0\}$ and

$$M_{\mathcal{L}} := \sum_{(i,j) \in I(\mathcal{L})} \mathbb{F} E_{ij},$$

where

$$I(\mathcal{L}) := \{(i, j) \in [n] \times [n] : (i, j) \preceq (i_\ell, j_\ell) \text{ for some } \ell \in [s]\},$$

and $E_{ij} \in M_n$ denotes the (i, j) standard matrix that has 1 as the (i, j) entry and 0 elsewhere.

In other words, $M_{\mathcal{L}}$ consists of matrices that have nonzero entries only in the upper right direction of some corner points (i_ℓ, j_ℓ) of \mathcal{L} . In [2], Brice and Huang introduce the notion of ladder matrix and proved that $M_{\mathcal{L}} \cdot M_{\mathcal{L}'} = M_{\mathcal{L}''}$, where \mathcal{L} and \mathcal{L}' are two arbitrary ladders of size n , and \mathcal{L}'' is a ladder decided by \mathcal{L} and \mathcal{L}' . In particular, if \mathcal{L} is an upper triangular ladder (i.e., $i_\ell < j_{\ell+1}$ for $\ell \in [s-1]$, see Definition 2.2), then $M_{\mathcal{L}}$ is a matrix subalgebra of M_n . Naturally, $M_{\mathcal{L}}$ is a Lie subalgebra of M_n (aka $\mathfrak{gl}(n, \mathbb{F})$) with respect to the standard Lie bracket $[X, Y] = XY - YX$.

Typical examples of Lie algebras $M_{\mathcal{L}}$ include those of block upper triangular matrices and of strictly block upper triangular matrices, M_{pq} embedded in the upper right corner of M_n (when $p \leq n$ and $q \leq n$), and M_n itself. In 1957, Dixmier and Lister constructed a nilpotent

Lie algebra [4] to disprove the converse of a statement of Jacobson [6]: “a Lie algebra with a nonsingular derivation is nilpotent”; the corresponding derivation algebra is clearly embedded in a special nilpotent $M_{\mathcal{L}}$.

A derivation of Lie algebra \mathfrak{g} is a linear map $f \in \text{End}(\mathfrak{g})$ that satisfies

$$f([X, Y]) = [f(X), Y] + [X, f(Y)] \quad \text{for all } X, Y \in \mathfrak{g}.$$

The Lie derivations and generalized derivations of ladder shape matrix Lie algebras over a field or ring has drawn much attention in recent years. Here is a fairly imcompleted list of literatures. Chen determines the structure of certain generalized derivations of a parabolic subalgebra of $\mathfrak{gl}(n, \mathbb{F})$ over a field \mathbb{F} with $\text{char}(\mathbb{F}) \neq 2$ and $|\mathbb{F}| > n \geq 3$ [9]. Brice describes the derviations of parabolic subalgebra of a reductive Lie algebra over an algebraically closed and characteristics zero field, and proves the zero-product determined property of such derivation algebras [1]. Let R be a communicative ring with identity. Cheung characterizes proper Lie derivations and gives sufficient conditions for any Lie derivation to be proper for triangular algebras over R [3]. Du and Wang investigate the Lie derivations of 2×2 block generalized matrix algebras [5]. Wang, Ou, and Yu describe the derivations of intermediate Lie algebras between diagonal matrix algebra and upper triangular matrix algebra in $\mathfrak{gl}(n, R)$ [10]. Wang and Yu characterized all the derivations of parabolic subalgebras of $\mathfrak{gl}(n, R)$ [8]. Ou, Wang, and Yao describe the derivations of the Lie algebra of stictly upper triangular matrices in $\mathfrak{gl}(n, R)$ [7]. Ji, Yang, and Chen study the biderivations of the algebra of strictly upper triangular matrices in $\mathfrak{gl}(n, R)$ [12]. The Lie triple derivations are also extensively studied, for examples, on $\mathfrak{gl}(n, R)$ [13], on the algebra of upper triangular matrices of $\mathfrak{gl}(n, R)$ [14], and on the parabolic subalgebras of $\mathfrak{gl}(n, R)$ [11].

In this paper, we explicitly characterize the derivations of the Lie algebra $M_{\mathcal{L}}$ associate with a dominant upper triangular (DUT) ladder \mathcal{L} for $\text{char}(\mathbb{F}) \neq 2$ (Theorem 3.1), and the derivations of $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ associate with a strongly dominant upper triangular (SDUT) ladder \mathcal{L} for $\text{char}(\mathbb{F}) \neq 2, 3$ (Theorem 5.3). A ladder $\mathcal{L} = \{(i_1, j_1), \dots, (i_s, j_s)\}$ is called DUT (resp. SDUT) if $j_\ell \leq i_\ell < j_{\ell+1}$ (resp. $j_\ell < i_\ell < j_{\ell+1}$) for $\ell \in [s-1]$. All $M_{\mathcal{L}}$ associate with a DUT ladder \mathcal{L} are completely characterized in Theorem 2.4.

- Theorem 2.4: \mathcal{L} is DUT if and only if $M_{\mathcal{L}}$ can be obtained by removing some non-consecutive diagonal blocks from the set of block upper triangular matrices corresponding to a partition of $[n]$.
- Theorem 3.1: When $\text{char}(\mathbb{F}) \neq 2$ and \mathcal{L} is a DUT ladder, every derivation of $M_{\mathcal{L}}$ is a sum of the adjoint action of a block upper triangular matrix and a linear map from $M_{\mathcal{L}}/[M_{\mathcal{L}}, M_{\mathcal{L}}]$ to the center of $M_{\mathcal{L}}$.
- Theorem 5.3: When $\text{char}(\mathbb{F}) \neq 2, 3$ and \mathcal{L} is a SDUT ladder, every derivation of $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ is the adjoint action of a block upper triangular matrix, so that it could be extended to a derivation of $M_{\mathcal{L}}$.

In general, a derivation of a Lie algebra stabilizes each subalgebra appearing in the derived series. Moreover, the derived series of a non-solvable Lie algebra of upper triangular ladder matrices will terminate at $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ for certain SDUT ladder \mathcal{L} . Therefore, knowledge on the derivation algebra of these $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ would be useful to disclose the structure of derivations of Lie algebras of general upper triangular ladder matrices.

The paper is organized as follow: Section 2 provides some basic properties of ladder matrices; in particular, all DUT ladder matrix algebras are completely characterized (Theorem

2.4), and the counting of these algebras in M_n is done (Corollary 2.5). Section 3 characterizes the derivations of $M_{\mathcal{L}}$ for DUT ladders \mathcal{L} and $\text{char}(\mathbb{F}) \neq 2$ (Theorem 3.1), and gives examples and applications, e.g. on the derivations of step 1 ladder matrix algebras (Theorem 3.5). Section 4 gives the proof of the main theorem in Section 3. Section 5 determines the derivations of $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ for SDUT ladders \mathcal{L} and $\text{char}(\mathbb{F}) \neq 2, 3$.

2. PRELIMINARY

We develop some basic properties of ladders and ladder matrices in this section. Given a ladder $\mathcal{L} \subset [n] \times [n]$, the matrices in $M_{\mathcal{L}}$ could be viewed as block matrices with respect to suitable partitions. A *partition* of $[n]$ can be characterized by a subset

$$\gamma = \{i_1, i_2, \dots, i_s\} \subseteq [n-1], \quad i_1 \leq i_2 \leq \dots \leq i_s,$$

where the corresponding partition in M_n is done *right after* the i_1, i_2, \dots, i_s rows and columns. Every ladder \mathcal{L} corresponds to one simplest compatible partition defined below.

Definition 2.1. Let $\mathcal{L} = \{(i_1, j_1), \dots, (i_s, j_s)\} \subset [n] \times [n]$ be a ladder. The partition of \mathcal{L} is a partition of $[n]$ characterized by

$$\gamma_{\mathcal{L}} := \{i_1, i_2, \dots, i_s\} \cup \{j_1 - 1, j_2 - 1, \dots, j_s - 1\} - \{0, n\}.$$

The matrices in $M_{\mathcal{L}}$ could be viewed as block matrices with respect to the partition $\gamma_{\mathcal{L}}$. Denote by $[I(\mathcal{L})]$ the index set of nonzero blocks of $M_{\mathcal{L}}$ with respect to $\gamma_{\mathcal{L}}$. If we set $t := |\gamma_{\mathcal{L}}| + 1$, then $[I(\mathcal{L})] \subset [t] \times [t]$.

The set of block upper triangular matrices corresponding to a partition $\gamma_{\mathcal{L}} = \{n_1, \dots, n_{t-1}\}$ is exactly $M_{\mathcal{L}_B}$, where

$$\mathcal{L}_B = \{(n_1, 1), (n_2, n_1 + 1), \dots, (n_{t-1}, n_{t-2} + 1), (n, n_{t-1} + 1)\}. \quad (2.1)$$

We introduce some special ladders to be used in the paper.

Definition 2.2. A ladder $\mathcal{L} = \{(i_1, j_1), \dots, (i_s, j_s)\}$ in $[n] \times [n]$ is called

- upper triangular: if $i_{\ell} < j_{\ell+1}$ for $\ell \in [s-1]$;
- strictly upper triangular: if $i_{\ell} < j_{\ell}$ for $\ell \in [s]$;
- dominant upper triangular (DUT): if $j_{\ell} \leq i_{\ell} < j_{\ell+1}$ for $\ell \in [s-1]$;
- strongly dominant upper triangular (SDUT): if $j_{\ell} < i_{\ell} < j_{\ell+1}$ for $\ell \in [s-1]$.

When \mathcal{L} is upper triangular, a matrix in $M_{\mathcal{L}}$ is called an upper triangular (\mathcal{L} -) ladder matrix. Similarly for the others.

The above different kinds of ladder \mathcal{L} can be easily distinguished by the shape of $M_{\mathcal{L}}$. They can also be reinterpreted by the block form of $M_{\mathcal{L}}$ with respect to the partition $\gamma_{\mathcal{L}}$:

- \mathcal{L} is upper triangular if $M_{\mathcal{L}} \subseteq M_{\mathcal{L}_B}$ (resp. $I(\mathcal{L}) \subseteq I(\mathcal{L}_B)$);
- \mathcal{L} is strictly upper triangular if $M_{\mathcal{L}}$ is contained in the strictly block upper triangular part of $M_{\mathcal{L}_B}$;
- \mathcal{L} is DUT if every block index $(i, j) \in [I(\mathcal{L})]$ is dominated by a diagonal one $(k, k) \in [I(\mathcal{L})]$;
- \mathcal{L} is SDUT if \mathcal{L} is DUT, and every nonzero diagonal block in $M_{\mathcal{L}}$ has size greater than 1.

Example 2.3. Consider the ladder $\mathcal{L} = \{(1, 1), (4, 3), (5, 5)\}$ of size 7. Then \mathcal{L} is DUT but not SDUT. The matrix form of $M_{\mathcal{L}}$ is given in Figure 1(a). The index set $I(\mathcal{L})$ of \mathcal{L} consists of $(i, j) \in [7] \times [7]$ dominated by at least one of $(1, 1)$, $(4, 3)$, and $(5, 5)$:

$$I(\mathcal{L}) = \{(1, j) : 1 \leq j \leq 7\} \cup \{(i, j) : 2 \leq i \leq 4, 3 \leq j \leq 7\} \cup \{(5, j) : 5 \leq j \leq 7\}.$$

The partition of \mathcal{L} is given by

$$\gamma_{\mathcal{L}} = \{1, 4, 5\} \cup \{1 - 1, 3 - 1, 5 - 1\} - \{0, 7\} = \{1, 2, 4, 5\}.$$

So matrices in $M_{\mathcal{L}}$ are partitioned after the 1, 2, 4, 5 rows and columns. Figure 1(b) indicates the block form of $M_{\mathcal{L}}$. The block index set $[I(\mathcal{L})]$ consists of $(i, j) \in [5] \times [5]$ dominated by at least one of $(1, 1)$, $(3, 3)$, and $(4, 4)$:

$$[I(\mathcal{L})] = \{(1, j) : 1 \leq j \leq 5\} \cup \{(i, j) : 2 \leq i \leq 3, 3 \leq j \leq 5\} \cup \{(4, 4), (4, 5)\}.$$

*	*	*	*	*	*	*
0	0	*	*	*	*	*
0	0	*	*	*	*	*
0	0	*	*	*	*	*
0	0	0	0	*	*	*
0	0	0	0	0	0	0
0	0	0	0	0	0	0

(a) matrix form of $M_{\mathcal{L}}$

*	*	*	*	*	*	*
0	0	*	*	*	*	*
0	0	*	*	*	*	*
0	0	*	*	*	*	*
0	0	0	0	*	*	*
0	0	0	0	0	0	0
0	0	0	0	0	0	0

(b) block matrix form of $M_{\mathcal{L}}$

FIGURE 1. Ladder $\mathcal{L} = \{(1, 1), (4, 3), (5, 5)\}$ of size 7

Now we can completely characterize DUT ladders and ladder matrices in terms of the associated partition.

Theorem 2.4. (1) A ladder \mathcal{L} is DUT if and only if for each $i \in [t - 1]$, at least one of (i, i) and $(i + 1, i + 1)$ is in $[I(\mathcal{L})]$. In particular, \mathcal{L} is DUT implies that $(i, j) \in [I(\mathcal{L})]$ for any $i, j \in [t]$ with $i < j$.
(2) Equivalently, $M_{\mathcal{L}}$ is a set of DUT ladder matrices, if and only if it can be obtained by removing some non-consecutive diagonal blocks from the set of block upper triangular matrices corresponding to a partition of $[n]$.

Proof. It suffices to prove the first statement. Let $\mathcal{L} = \{(i_1, j_1), \dots, (i_s, j_s)\}$ and $\gamma_{\mathcal{L}} = \{n_1, \dots, n_{t-1}\}$, so that the set of block upper triangular matrices is $M_{\mathcal{L}_B}$ for the ladder $\mathcal{L}_B = \{(n_1, 1), (n_2, n_1 + 1), \dots, (n_{t-1}, n_{t-2} + 1), (n, n_{t-1} + 1)\}$.

Suppose for each $i \in [t - 1]$, at least one of (i, i) and $(i + 1, i + 1)$ is in $[I(\mathcal{L})]$. Then $\mathcal{L} \subseteq \mathcal{L}_B$. It is obvious that $j_{\ell} \leq i_{\ell} < j_{\ell+1}$ for $\ell \in [s - 1]$. Hence \mathcal{L} is DUT.

Now assume that \mathcal{L} is DUT. Then every block index $(i, j) \in [I(\mathcal{L})]$ is dominated by a diagonal block index $(k, k) \in [I(\mathcal{L})]$, that is, $i \leq k \leq j$. If for some $i \in [t - 1]$, neither (i, i) nor $(i + 1, i + 1)$ is in $[I(\mathcal{L})]$, then $(i, i + 1)$ is not in $[I(\mathcal{L})]$. Then $n_i \notin \gamma_{\mathcal{L}}$, which is a contradiction. Therefore, at least one of (i, i) and $(i + 1, i + 1)$ is in $[I(\mathcal{L})]$. \square

A direct application of Theorem 2.4 is the counting of DUT ladder matrices.

Corollary 2.5. Let $\{F_t\}_{t=1}^{\infty} = \{1, 1, 2, 3, 5, \dots\}$ be the Fibonacci sequence.

- (1) The number of sets of DUT ladder matrices corresponding to a $t \times t$ block form equals to b_t , where

$$\{b_t\}_{t=1}^{\infty} = \{F_{t+2}\}_{t=1}^{\infty} = \{2, 3, 5, 8, 13, \dots\}. \quad (2.2)$$

- (2) The number of sets of DUT ladder matrices in M_n equals to a_n , where

$$\{a_n\}_{n=1}^{\infty} = \{F_{2n+1}\}_{n=1}^{\infty} = \{2, 5, 13, 34, 89, \dots\}. \quad (2.3)$$

Proof. (1) Clearly $b_1 = 2 = F_3$ and $b_2 = 3 = F_4$. (2.2) will be proved if $\{b_t\}$ satisfies the same recursive formula as $\{F_{t+2}\}$ does, that is,

$$b_t = b_{t-1} + b_{t-2}. \quad (2.4)$$

By Theorem 2.4, b_t equals to the number of ways to choose non-consecutive diagonal blocks in a given $t \times t$ block form. If the first diagonal block is chosen, then the second one should be skipped, and there are b_{t-2} ways to choose the remaining diagonal blocks; if the first diagonal block is not chosen, then there are b_{t-1} ways to choose the remaining diagonal blocks. Therefore, (2.4) is true and (2.2) is proved.

- (2) Given $t \in [n]$, there are $\binom{n-1}{t-1}$ ways to partition matrices in M_n into a $t \times t$ block form; each block form corresponds to $b_t = F_{t+2}$ sets of DUT ladder matrices. Let $r_1 := \frac{1+\sqrt{5}}{2}$ and $r_2 := \frac{1-\sqrt{5}}{2}$ be the roots of $x^2 - x - 1 = 0$. The Binet's Fibonacci number formula says that

$$F_t = \frac{1}{\sqrt{5}} r_1^t - \frac{1}{\sqrt{5}} r_2^t.$$

Therefore,

$$\begin{aligned} a_n &= \sum_{t=1}^n \binom{n-1}{t-1} F_{t+2} = \sum_{t=1}^n \binom{n-1}{t-1} \left(\frac{1}{\sqrt{5}} r_1^{t+2} - \frac{1}{\sqrt{5}} r_2^{t+2} \right) \\ &= \frac{1}{\sqrt{5}} [r_1^3(1+r_1)^{n-1} - r_2^3(1+r_2)^{n-1}] = \frac{1}{\sqrt{5}} [r_1^3(r_1^2)^{n-1} - r_2^3(r_2^2)^{n-1}] = F_{2n+1}. \quad \square \end{aligned}$$

We give some notations that will be used in studying the partitioned matrices associated with $M_{\mathcal{L}}$.

Definition 2.6. Given an algebra $(M, +, *)$ and two subsets $M', M'' \subseteq M$, define the subset

$$M' * M'' := \left\{ \sum_{i=1}^m A_i * B_i \mid m \in \mathbb{N}, A_i \in M', B_i \in M'' \right\}.$$

Definition 2.7. Consider the matrices in M_n with respect to a given partition $\gamma_{\mathcal{L}}$.

- Let \mathcal{M}_{ij} denote the set of all submatrices in the (i, j) block of M_n . Let $E_{pq}^{[ij]}$ denote the (p, q) standard matrix in \mathcal{M}_{ij} .
- Let $\widetilde{\mathcal{M}}_{ij}$ denote the embedding of \mathcal{M}_{ij} in M_n .
- For $A \in M_n$, let $A_{ij} \in \mathcal{M}_{ij}$ denote the (i, j) block submatrix of A .
- For $B_{ij} \in \mathcal{M}_{ij}$, let $\widetilde{B}_{ij} \in \widetilde{\mathcal{M}}_{ij}$ denote the matrix in M_n with B_{ij} in the (i, j) block and zero elsewhere. Similarly for $\widetilde{B}_{ik} \widetilde{B}_{kj}$ if $B_{ik} \in \mathcal{M}_{ik}$ and $B_{kj} \in \mathcal{M}_{kj}$.
- In \mathcal{M}_{kk} , let I_{kk} denote the identity matrix, and \mathfrak{sl}_{kk} the set of traceless matrices, respectively.

A notation of double index, say \mathcal{M}_{ij} , may be written as $\mathcal{M}_{i,j}$ for clarity purpose.

The *normalizer* $N(M_{\mathcal{L}})$ and the *centralizer* $Z(M_{\mathcal{L}})$ of Lie subalgebra $M_{\mathcal{L}}$ in M_n are:

$$\begin{aligned} N(M_{\mathcal{L}}) &= \{A \in M_n : [A, B] \in M_{\mathcal{L}} \text{ for all } B \in M_{\mathcal{L}}\}, \\ Z(M_{\mathcal{L}}) &= \{A \in M_n : [A, B] = 0 \text{ for all } B \in M_{\mathcal{L}}\}. \end{aligned}$$

They are explicitly described by the following two lemmas.

Lemma 2.8. *If \mathcal{L} is a DUT ladder, then $N(M_{\mathcal{L}}) = M_{\mathcal{L}_B}$, the subalgebra of block upper triangular matrices with respect to the partition of \mathcal{L} .*

Proof. We first show that $N(M_{\mathcal{L}}) \subseteq M_{\mathcal{L}_B}$. Suppose on the contrary, there is $A \in N(M_{\mathcal{L}})$ such that the (i, j) block $A_{ij} \neq 0$ for some $i > j$. There are two cases:

- (1) $i > j + 1$: We have $\widetilde{\mathcal{M}_{j,j+1}} \subseteq M_{\mathcal{L}}$ by Theorem 2.4. So $[A, \widetilde{\mathcal{M}_{j,j+1}}] \subseteq M_{\mathcal{L}}$. However, its $(i, j + 1)$ block is

$$[A, \widetilde{\mathcal{M}_{j,j+1}}]_{i,j+1} = [A_{ij}, \mathcal{M}_{j,j+1}] = A_{ij}\mathcal{M}_{j,j+1} \neq \{0\},$$

which contradicts to the DUT assumption of \mathcal{L} .

- (2) $i = j + 1$: By Theorem 2.4, either $\widetilde{\mathcal{M}_{jj}} \subseteq M_{\mathcal{L}}$ or $\widetilde{\mathcal{M}_{j+1,j+1}} \subseteq M_{\mathcal{L}}$. Without loss of generality, suppose $\widetilde{\mathcal{M}_{jj}} \subseteq M_{\mathcal{L}}$. Then $[A, \widetilde{\mathcal{M}_{jj}}] \subseteq M_{\mathcal{L}}$. However, its (i, j) block is

$$[A, \widetilde{\mathcal{M}_{jj}}]_{ij} = A_{ij}\mathcal{M}_{jj} \neq \{0\},$$

which contradicts the DUT assumption of \mathcal{L} .

Therefore, $A \in M_{\mathcal{L}_B}$ and thus $N(M_{\mathcal{L}}) \subseteq M_{\mathcal{L}_B}$.

For any $(i, j) \in [t] \times [t]$ with $i \leq j$, the possibly nonzero blocks of matrices in $[\widetilde{\mathcal{M}_{ij}}, M_{\mathcal{L}}]$ are those (i, q) blocks with $q \geq j$ and (p, j) blocks with $p \leq i$, all of which belong to $M_{\mathcal{L}}$. Hence $M_{\mathcal{L}_B} \subseteq N(M_{\mathcal{L}})$. \square

Lemma 2.9. *Let \mathcal{L} be a DUT ladder and $t = |\gamma_{\mathcal{L}}| + 1$.*

- (1) *If both the $(1, 1)$ and the (t, t) blocks of $M_{\mathcal{L}}$ are zero, then $Z(M_{\mathcal{L}}) = \mathbb{F}I_n + \widetilde{\mathcal{M}_{1t}}$.*
(2) *Otherwise, $Z(M_{\mathcal{L}}) = \mathbb{F}I_n$.*

Proof. Clearly $Z(M_{\mathcal{L}}) \subseteq N(M_{\mathcal{L}})$. The possibly nonzero blocks of any $A \in Z(M_{\mathcal{L}})$ are A_{ij} for some $1 \leq i \leq j \leq t$. If $A_{ij} \neq 0$ and $2 \leq i < j$, then $\widetilde{\mathcal{M}_{i-1,i}} \subseteq M_{\mathcal{L}}$, and we can find $B_{i-1,i} \in \mathcal{M}_{i-1,i}$ such that

$$0 \neq B_{i-1,i}A_{i,j} = [B_{i-1,i}, A_{i,j}] = [\widetilde{B_{i-1,i}}, A]_{i-1,j},$$

which contradicts to the assumption $A \in Z(M_{\mathcal{L}})$. Thus $A_{ij} = 0$ for all $2 \leq i < j \leq t$. Similarly, $A_{ij} = 0$ for all $1 \leq i < j \leq t - 1$. So the only possibly nonzero blocks of $A \in Z(M_{\mathcal{L}})$ are A_{1t} and A_{ii} for $i \in [t]$.

If $(1, 1) \in [I(\mathcal{L})]$, then $0 = [I_{11}, A]_{1t} = A_{1t}$. Similarly, $(t, t) \in [I(\mathcal{L})]$ implies that $A_{1t} = 0$. If neither $(1, 1)$ nor (t, t) is in $[I(\mathcal{L})]$, then $\widetilde{\mathcal{M}_{1t}}$ is in $Z(M_{\mathcal{L}})$ by direct computation.

Now for any $i, j \in [t]$ with $i < j$ and $\widetilde{B_{ij}} \in \widetilde{\mathcal{M}_{ij}} \subseteq M_{\mathcal{L}}$,

$$0 = [A, \widetilde{B_{ij}}]_{ij} = A_{ii}B_{ij} - B_{ij}A_{jj}.$$

Let B_{ij} go through all standard matrices in \mathcal{M}_{ij} that have an entry one and zeros elsewhere. We can get $A_{ii} = \lambda I_{ii}$ and $A_{jj} = \lambda I_{jj}$ for a fixed $\lambda \in \mathbb{F}$.

In summary, $Z(M_{\mathcal{L}})$ is described by the statements (1) and (2). \square

3. THE MAIN THEOREM

In this section, we explicitly characterize the derivation algebra $\text{Der}(M_{\mathcal{L}})$ for any DUT ladder \mathcal{L} over a field \mathbb{F} with $\text{char}(\mathbb{F}) \neq 2$, and provide some consequent results. Note that the adjoint representation $\text{ad} : M_n \rightarrow \text{Der}(M_n)$ defined by $\text{ad } A(B) = [A, B]$ induces a Lie algebra homomorphism

$$\text{ad}(\cdot)|_{M_{\mathcal{L}}} : \text{N}(M_{\mathcal{L}})/\text{Z}(M_{\mathcal{L}}) \rightarrow \text{Der}(M_{\mathcal{L}}),$$

which will be used in the following theorem.

Theorem 3.1. (Main theorem) Suppose $\text{char}(\mathbb{F}) \neq 2$. Let \mathcal{L} be a DUT ladder. Then the Lie algebra $\text{Der}(M_{\mathcal{L}})$ can be decomposed as a direct sum of ideals:

$$\text{Der}(M_{\mathcal{L}}) = \text{ad}(\text{N}(M_{\mathcal{L}})/\text{Z}(M_{\mathcal{L}}))|_{M_{\mathcal{L}}} \oplus \mathcal{D} \quad (3.1)$$

$$= \left(\text{ad} \left(\frac{M_{\mathcal{L}}}{\text{Z}(M_{\mathcal{L}}) \cap M_{\mathcal{L}}} \right) \rtimes \bigoplus_{(k,k) \in [I(\mathcal{L}_B)] - [I(\mathcal{L})]} \text{ad}(\widetilde{\mathcal{M}}_{kk}) \right) \Big|_{M_{\mathcal{L}}} \oplus \mathcal{D} \quad (3.2)$$

where

- the normalizer $\text{N}(M_{\mathcal{L}})$ and the centralizer $\text{Z}(M_{\mathcal{L}})$ are described by Lemmas 2.8 and 2.9, respectively;
- the ideal \mathcal{D} is defined by

$$\mathcal{D} := \{\phi \in \text{End}(M_{\mathcal{L}}) : \text{Ker } \phi \supseteq [M_{\mathcal{L}}, M_{\mathcal{L}}], \text{ Im } \phi \subseteq \text{Z}(M_{\mathcal{L}}) \cap M_{\mathcal{L}}\}; \quad (3.3)$$

in particular, $\mathcal{D} \simeq \text{Hom}_{\mathbb{F}}(M_{\mathcal{L}}/[M_{\mathcal{L}}, M_{\mathcal{L}}], \text{Z}(M_{\mathcal{L}}) \cap M_{\mathcal{L}})$ as vector spaces.

Explicitly, we have the following cases with respect to the partition $\gamma_{\mathcal{L}}$ of \mathcal{L} (let $t = |\gamma_{\mathcal{L}}| + 1$):

- (1) If $M_{\mathcal{L}}$ is a set of block upper triangular matrices (i.e. $\mathcal{L} = \mathcal{L}_B$), then every $f \in \text{Der}(M_{\mathcal{L}})$ corresponds to an $X \in M_{\mathcal{L}_B}/\mathbb{F}I_n$ and $c_1, \dots, c_t \in \mathbb{F}$, such that

$$f(A) = \text{ad } X(A) + \left(\sum_{k \in [t]} c_k \text{tr}(A_{kk}) \right) I_n \quad \text{for } A \in M_{\mathcal{L}}. \quad (3.4)$$

- (2) If $M_{\mathcal{L}}$ has some zero diagonal block(s), but at least one of its $(1, 1)$ and (t, t) blocks is nonzero, then every $f \in \text{Der}(M_{\mathcal{L}})$ corresponds to an $X \in M_{\mathcal{L}_B}/\mathbb{F}I_n$, such that

$$f(A) = \text{ad } X(A) \quad \text{for } A \in M_{\mathcal{L}}. \quad (3.5)$$

- (3) If both the $(1, 1)$ and the (t, t) blocks in $M_{\mathcal{L}}$ are zero, then every $f \in \text{Der}(M_{\mathcal{L}})$ corresponds to an $X \in M_{\mathcal{L}_B}/(\mathbb{F}I_n + \widetilde{\mathcal{M}}_{1t})$ and $Y_{1tk} \in \mathcal{M}_{1t}$ for each $(k, k) \in [I(\mathcal{L})]$, such that

$$f(A) = \text{ad } X(A) + \sum_{(k,k) \in [I(\mathcal{L})]} \text{tr}(A_{kk}) \widetilde{Y}_{1tk} \quad \text{for } A \in M_{\mathcal{L}}. \quad (3.6)$$

A detailed proof of Theorem 3.1 will be given in Section 4. The special case $\mathcal{L} = \mathcal{L}_B$ (where $M_{\mathcal{L}}$ is a set of block upper triangular matrices) is included in a paper of Dengyin Wang and Qiu Yu [8, Theorem 4.1]. Moreover, Daniel Brice obtains a formula similar to (3.1) for the derivation algebra of the parabolic subalgebra of a reductive Lie algebra over a \mathbb{C} -like fields or over \mathbb{R} [1].

Example 3.2. *Theorem 3.1 is not true when $\text{char}(\mathbb{F}) = 2$. Consider $M_{\mathcal{L}} = M_2$ with the basis $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Define $f \in \text{End}(M_{\mathcal{L}})$ by $f(E_{12}) = E_{21}$ and $f(E_{ij}) = 0$ for $(i, j) = (1, 1), (2, 1), (2, 2)$. It is straightforward to verify that*

$$f([E, E']) = [f(E), E'] + [E, f(E')] \quad (3.7)$$

for any $E, E' \in \mathcal{B}$, since there are only two cases that either side of (3.7) is nonzero: $\{E, E'\} = \{E_{11}, E_{12}\}$ or $\{E_{12}, E_{22}\}$. Therefore $f \in \text{Der}(M_{\mathcal{L}})$. However, f is not an element of $\text{ad}(\text{N}(M_{\mathcal{L}})/\text{Z}(M_{\mathcal{L}}))|_{M_{\mathcal{L}}} \oplus \mathcal{D}$ in (3.1).

When \mathcal{L} is an upper triangular ladder, an inner derivation $(\text{ad } X)|_{M_{\mathcal{L}}}$ ($X \in M_{\mathcal{L}}$) satisfies that

$$(\text{ad } X)|_{M_{\mathcal{L}}}(\widetilde{\mathcal{M}}_{ij}) \subseteq \widetilde{\mathcal{M}}_{ij} + \sum_{k>j} \widetilde{\mathcal{M}}_{ik} + \sum_{\ell<i} \widetilde{\mathcal{M}}_{\ell j} \quad \text{for any } \widetilde{\mathcal{M}}_{ij} \subseteq M_{\mathcal{L}}.$$

So the inner derivation sends the (i, j) block to a sum of blocks with the indices dominated by (i, j) . This dominance property also holds for all derivations of $M_{\mathcal{L}}$ when \mathcal{L} is a DUT ladder with some zero diagonal blocks (i.e. $M_{\mathcal{L}} \neq M_{\mathcal{L}_B}$).

Corollary 3.3. *Let \mathcal{L} be a DUT ladder with some zero diagonal blocks. Then every $f \in \text{Der}(M_{\mathcal{L}})$ maps any (i, j) block of $M_{\mathcal{L}}$ to a sum of some blocks dominated by the (i, j) block.*

Proof. The corollary is a direct consequence of Theorem 3.1(2) and (3). \square

In general, Corollary 3.3 may not be true if \mathcal{L} is not a DUT ladder which can be seen via the following example.

Example 3.4. *Suppose \mathbb{F} is an arbitrary field. Let $n = 5$ and $\mathcal{L} = \{(1, 2), (3, 4)\}$. Then \mathcal{L} is not DUT, and $M_{\mathcal{L}}$ has the form:*

$$\left(\begin{array}{c|cc|cc} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ \hline 0 & 0 & 0 & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad a_{ij} \in \mathbb{F}.$$

So $M_{\mathcal{L}}$ has a basis $\mathcal{B} = \{E_{12}, E_{13}, E_{14}, E_{15}, E_{24}, E_{25}, E_{34}, E_{35}\}$. Given $a, b \in \mathbb{F}$, define $f \in \text{End}(M_{\mathcal{L}})$ by

$$f(E_{12}) := \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad f(E_{13}) := \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

and $f(E) = 0$ for all other matrices E in the basis \mathcal{B} . We prove that

$$f([E, E']) = [f(E), E'] + [E, f(E')] \quad \text{for all } E, E' \in \mathcal{B}, \quad (3.8)$$

so that f is a derivation of $M_{\mathcal{L}}$. On one hand, $[E, E'] \in \text{span}\{E_{14}, E_{15}\}$ and thus $f([E, E']) = 0$; on the other hand, in (3.8), $[f(E), E'] \neq 0$ or $[E, f(E')] \neq 0$ only when $\{E, E'\} = \{E_{12}, E_{13}\}$, for which the equality (3.8) is easily verified. Therefore, $f \in \text{Der}(M_{\mathcal{L}})$. However, f maps the block $\widetilde{\mathcal{M}}_{12}$ into $\widetilde{\mathcal{M}}_{23}$, where $(2, 3)$ is not dominated by $(1, 2)$.

An important family of ladders is that of 1-step ladders $\mathcal{L} = \{(i, j)\}$, where each $M_{\mathcal{L}}$ realizes M_{pq} ($p, q \leq n$) as a Lie subalgebra of M_n . Many 1-step ladders are DUT. The derivations of these $M_{\mathcal{L}}$ can be explicitly characterized here.

Theorem 3.5. *Let $\mathcal{L} = \{(i, j)\} \subseteq [n] \times [n]$ be a 1-step ladder of size n .*

(1) *If $i < j$, then $M_{\mathcal{L}}$ is abelian and*

$$\text{Der}(M_{\mathcal{L}}) = \text{End}(M_{\mathcal{L}}).$$

(2) *If $i = n$ or $j = 1$, then*

$$\text{Der}(M_{\mathcal{L}}) = \text{ad}(\text{N}(M_{\mathcal{L}})/\text{Z}(M_{\mathcal{L}}))|_{M_{\mathcal{L}}}.$$

Explicitly, there are three subcases:

(a) *If $i = n$ and $j = 1$, then $M_{\mathcal{L}} = M_n$, and $\text{Der}(M_{\mathcal{L}}) = \text{ad}(M_n/\mathbb{F}I_n)$.*

(b) *If $i \neq n$ and $j = 1$, then*

$$\begin{aligned} M_{\mathcal{L}} &= \left\{ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \in M_n : A_{11} \in M_i, A_{12} \in M_{i, n-i} \right\}, \\ \text{Der}(M_{\mathcal{L}}) &= \text{ad} \left\{ \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} : X_{11} \in M_i, X_{12} \in M_{i, n-i}, X_{22} \in M_{n-i} \right\} \Big|_{M_{\mathcal{L}}} \\ &= \text{ad } M_{\mathcal{L}} \rtimes \text{ad}(\widetilde{\mathcal{M}}_{22})|_{M_{\mathcal{L}}}. \end{aligned}$$

(c) *If $i = n$ and $j \neq 1$, then*

$$\begin{aligned} M_{\mathcal{L}} &= \left\{ \begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix} \in M_n : A_{12} \in M_{j-1, n-j+1}, A_{22} \in M_{n-j+1} \right\}, \\ \text{Der}(M_{\mathcal{L}}) &= \text{ad} \left\{ \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} : X_{11} \in M_{j-1}, X_{12} \in M_{j-1, n-j+1}, X_{22} \in M_{n-j+1} \right\} \Big|_{M_{\mathcal{L}}} \\ &= \text{ad } M_{\mathcal{L}} \rtimes \text{ad}(\widetilde{\mathcal{M}}_{11})|_{M_{\mathcal{L}}}. \end{aligned}$$

(3) *If $n > i \geq j > 1$. Then*

$$\text{Der}(M_{\mathcal{L}}) = \text{ad}(\text{N}(M_{\mathcal{L}})/\text{Z}(M_{\mathcal{L}}))|_{M_{\mathcal{L}}} \oplus \mathcal{D}$$

where \mathcal{D} is defined in (3.3). Explicitly, $\gamma_{\mathcal{L}} = \{j-1, i\}$, and

$$\begin{aligned} M_{\mathcal{L}} &= \left\{ A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & 0 \end{bmatrix} \right\}, \\ \text{Der}(M_{\mathcal{L}}) &= \text{ad} \left\{ \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} \right\} \Big|_{M_{\mathcal{L}}} \oplus \left\{ f_Y : f_Y(A) = \text{tr}(A_{22}) \begin{bmatrix} 0 & 0 & Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \\ &= \left(\text{ad } M_{\mathcal{L}} \rtimes \left(\text{ad}(\widetilde{\mathcal{M}}_{11}) \oplus \text{ad}(\widetilde{\mathcal{M}}_{33}) \right) \Big|_{M_{\mathcal{L}}} \right) \oplus \left\{ f_Y : f_Y(A) = \text{tr}(A_{22}) \begin{bmatrix} 0 & 0 & Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}. \end{aligned}$$

Proof. The cases (2) and (3) are done by Theorem 3.1. For case (1) where $M_{\mathcal{L}}$ is abelian, every $f \in \text{End}(M_{\mathcal{L}})$ satisfies that

$$f([A, B]) = 0 = [f(A), B] + [A, f(B)], \quad A, B \in M_{\mathcal{L}}.$$

Therefore, $\text{Der}(M_{\mathcal{L}}) = \text{End}(M_{\mathcal{L}})$. □

4. PROOF OF THE MAIN THEOREM

To prove Theorem 3.1, we give several auxiliary results here. The first two lemmas below connect the linear transformations within the four blocks of a 2×2 block matrix:

$$\begin{array}{cc} p & q \\ m & \begin{bmatrix} M_{mp} & M_{mq} \\ M_{np} & M_{nq} \end{bmatrix} \\ n & \end{array}$$

Let $E_{ij}^{(mn)}$ denote the (i, j) standard matrix in M_{mn} .

Lemma 4.1. *Suppose \mathbb{F} is an arbitrary field. If linear transformations $\phi : M_{mp} \rightarrow M_{mq}$ and $\varphi : M_{np} \rightarrow M_{nq}$ satisfy that*

$$\phi(AB) = A\varphi(B) \quad \text{for all } A \in M_{mn}, B \in M_{np}, \quad (4.1)$$

then there is $X \in M_{pq}$ such that $\phi(C) = CX$ for $C \in M_{mp}$ and $\varphi(D) = DX$ for $D \in M_{np}$.

Proof. For any $j \in [n]$ and any $B \in M_{np}$,

$$\phi(E_{1j}^{(mn)}B) = E_{1j}^{(mn)}\varphi(B).$$

All such $E_{1j}^{(mn)}B$ span the first row space of M_{mp} . So ϕ maps the first row of M_{mp} to the first row of M_{mq} . There exists a unique $X \in M_{pq}$ such that

$$E_{1j}^{(mn)}\varphi(B) = \phi(E_{1j}^{(mn)}B) = E_{1j}^{(mn)}BX, \quad \text{for all } j \in [n], B \in M_{np}.$$

Therefore, $\varphi(B) = BX$. Then $\phi(AB) = A\varphi(B) = ABX$ for any $A \in M_{mn}$ and $B \in M_{np}$. Hence $\phi(C) = CX$ for all $C \in M_{mp}$. \square

Lemma 4.2. *Suppose \mathbb{F} is an arbitrary field. If linear transformations $\phi : M_{mp} \rightarrow M_{np}$ and $\varphi : M_{mq} \rightarrow M_{nq}$ satisfy that*

$$\phi(BA) = \varphi(B)A \quad \text{for all } A \in M_{qp}, B \in M_{mq}, \quad (4.2)$$

then there is $X \in M_{nm}$ such that $\phi(C) = XC$ for $C \in M_{mp}$ and $\varphi(D) = XD$ for $D \in M_{mq}$.

The proof (omitted) is similar to that of Lemma 4.1.

Lemma 4.3. *Suppose \mathbb{F} is an arbitrary field. If $X \in M_m$ and $Y \in M_n$ satisfy that $XA = AY$ for all $A \in M_{mn}$, then $X = \lambda I_m$ and $Y = \lambda I_n$ for certain $\lambda \in \mathbb{F}$.*

Proof. For any $(i, j) \in [m] \times [n]$,

$$XE_{ij} = E_{ij}Y.$$

Comparing the (i, j) entry, we get $x_{ii} = y_{jj}$. Comparing the (p, j) entry for $p \neq i$, we get $x_{pi} = 0$. Comparing the (i, q) entry for $q \neq j$, we get $0 = y_{jq}$. Therefore, $X = \lambda I_m$ and $Y = \lambda I_n$ for some $\lambda \in \mathbb{F}$. \square

In the remaining of this section, we assume that $\text{char}(\mathbb{F}) \neq 2$, \mathcal{L} is a DUT ladder, and $t := |\gamma_{\mathcal{L}}| + 1$. Next we present several results on the image of a derivation of $M_{\mathcal{L}}$.

Lemma 4.4. *For $f \in \text{Der}(M_{\mathcal{L}})$ and $(k, k) \in [I(\mathcal{L})]$, the f -images of the identity matrix and the standard matrices in the (k, k) block satisfy that*

$$f(\widetilde{I_{kk}}), f(\widetilde{E_{\ell\ell}^{[kk]}}) \in \sum_{i=1}^{k-1} \widetilde{\mathcal{M}_{ik}} + \sum_{j=k+1}^t \widetilde{\mathcal{M}_{kj}} + (Z(M_{\mathcal{L}}) \cap M_{\mathcal{L}}) \quad (4.3)$$

where (by Lemma 2.9)

$$Z(M_{\mathcal{L}}) \cap M_{\mathcal{L}} = \begin{cases} \mathbb{F}I_n & \text{if } \mathcal{L} = \mathcal{L}_B; \\ \widetilde{\mathcal{M}}_{1t} & \text{if } (1, 1) \notin [I(\mathcal{L})] \text{ and } (t, t) \notin [I(\mathcal{L})]; \\ 0 & \text{otherwise.} \end{cases} \quad (4.4)$$

Proof. We prove (4.3) for $f(\widetilde{I}_{kk})$ here, and the case of $f(\widetilde{E_{\ell\ell}^{[kk]}})$ is similar.

(1) First we investigate $f(\widetilde{I}_{kk})_{jj}$. When $k < j$,

$$\begin{aligned} f(\widetilde{A_{kj}})_{kj} &= f([\widetilde{I_{kk}}, \widetilde{A_{kj}}])_{kj} = [f(\widetilde{I_{kk}}), \widetilde{A_{kj}}]_{kj} + [\widetilde{I_{kk}}, f(\widetilde{A_{kj}})]_{kj} \\ &= f(\widetilde{I_{kk}})_{kk} A_{kj} - A_{kj} f(\widetilde{I_{kk}})_{jj} + f(\widetilde{A_{kj}})_{kj} \end{aligned}$$

Therefore

$$f(\widetilde{I_{kk}})_{kk} A_{kj} = A_{kj} f(\widetilde{I_{kk}})_{jj} \quad \text{for } A_{kj} \in \mathcal{M}_{kj}.$$

Lemma 4.3 implies that $f(\widetilde{I_{kk}})_{kk} = \lambda I_{kk}$ and $f(\widetilde{I_{kk}})_{jj} = \lambda I_{jj}$ for a $\lambda \in \mathbb{F}$. The same equation holds for $k > j$. In the situation $\mathcal{L} \neq \mathcal{L}_B$, there exists $(p, p) \notin [I(\mathcal{L})]$, which forces $f(\widetilde{I_{kk}})_{pp} = 0$ and thus $f(\widetilde{I_{kk}})_{jj} = 0$ for all $j \in [t]$.

(2) Next we prove that $f(\widetilde{I_{kk}})_{ij} = 0$ for $i < j$, $i \neq k$, $j \neq k$, and $(i, j) \neq (1, t)$. Either $i > 1$ or $j < t$. Without loss of generality, suppose $j < t$ (similarly for $i > 1$). Then

$$f([\widetilde{I_{kk}}, \widetilde{A_{jt}}])_{it} = [f(\widetilde{I_{kk}}), \widetilde{A_{jt}}]_{it} + [\widetilde{I_{kk}}, f(\widetilde{A_{jt}})]_{it}. \quad (4.5)$$

- (a) If $k \neq t$, then (4.5) becomes $0 = f(\widetilde{I_{kk}})_{ij} A_{jt}$ for any $A_{jt} \in \mathcal{M}_{jt}$. So $f(\widetilde{I_{kk}})_{ij} = 0$.
- (b) If $k = t$, then (4.5) becomes

$$-f(\widetilde{A_{jt}})_{it} = f(\widetilde{I_{kk}})_{ij} A_{jt} - f(\widetilde{A_{jt}})_{it}.$$

Again we get $0 = f(\widetilde{I_{kk}})_{ij} A_{jt}$ and thus $f(\widetilde{I_{kk}})_{ij} = 0$.

(3) Finally, if $(1, 1) \in [I(\mathcal{L})]$ or $(t, t) \in [I(\mathcal{L})]$, say $(1, 1) \in [I(\mathcal{L})]$, then for any $(k, k) \in [I(\mathcal{L})]$ and $k \notin \{1, t\}$,

$$0 = f([\widetilde{I_{11}}, \widetilde{I_{kk}}])_{1t} = [f(\widetilde{I_{11}}), \widetilde{I_{kk}}]_{1t} + [\widetilde{I_{11}}, f(\widetilde{I_{kk}})]_{1t} = f(\widetilde{I_{kk}})_{1t}.$$

Lemma 2.9 implies (4.4). Therefore, (4.3) is proved. \square

For $(p, q) \in [I(\mathcal{L})]$, we have

$$\widetilde{\mathcal{M}}_{pq} \cap [M_{\mathcal{L}}, M_{\mathcal{L}}] = \begin{cases} \widetilde{\mathfrak{sl}}_{pp}, & \text{if } p = q; \\ \widetilde{\mathcal{M}}_{pq}, & \text{if } p < q. \end{cases}$$

Next we investigate the image of derivations on each block in $[M_{\mathcal{L}}, M_{\mathcal{L}}]$.

Lemma 4.5. Suppose $\text{char}(\mathbb{F}) \neq 2$. For $f \in \text{Der}(M_{\mathcal{L}})$, $(p, q) \in [I(\mathcal{L})]$, and $\widetilde{A_{pq}} \in \widetilde{\mathcal{M}}_{pq} \cap [M_{\mathcal{L}}, M_{\mathcal{L}}]$,

$$f(\widetilde{A_{pq}}) \in \widetilde{\mathcal{M}}_{pq} + \sum_{i=1}^{p-1} \widetilde{\mathcal{M}}_{iq} + \sum_{j=q+1}^t \widetilde{\mathcal{M}}_{pj}. \quad (4.6)$$

Proof. There are two cases for $(p, q) \in [I(\mathcal{L})]$:

(1) $p = q$: Then $\widetilde{\mathcal{M}}_{pq} \cap [M_{\mathcal{L}}, M_{\mathcal{L}}] = \widetilde{\mathfrak{sl}}_{pp} = [\widetilde{\mathfrak{sl}}_{pp}, \widetilde{\mathfrak{sl}}_{pp}]$. For $B_{pp}, C_{pp} \in \mathfrak{sl}_{pp}$,

$$f([\widetilde{B_{pp}}, \widetilde{C_{pp}}]) = [f(\widetilde{B_{pp}}), \widetilde{C_{pp}}] + [\widetilde{B_{pp}}, f(\widetilde{C_{pp}})]. \quad (4.7)$$

Since $f(\widetilde{B_{pp}})$ and $f(\widetilde{C_{pp}})$ are block upper triangular matrices with respect to $\gamma_{\mathcal{L}}$, the nonzero (i, j) blocks of the right side of (4.7) satisfy that $p = i \leq j$ or $i \leq j = p$. Thus (4.6) holds in this case.

(2) $p < q$: Then $\widetilde{\mathcal{M}}_{pq} \cap [M_{\mathcal{L}}, M_{\mathcal{L}}] = \widetilde{\mathcal{M}}_{pq}$. Let $q = p + k$ and we prove (4.6) by induction on k . For better display, we also use $\{\cdot\}_{ij}$ here to denote the embedding of \mathcal{M}_{ij} to $\widetilde{\mathcal{M}}_{ij} \subseteq M_n$.

(a) $k = 1$: By Theorem 2.4, at least one of (p, p) and $(p + 1, p + 1)$ is in $[I(\mathcal{L})]$.

Without loss of generality, suppose $(p, p) \in [I(\mathcal{L})]$. Then for $A_{p,p+1} \in \mathcal{M}_{p,p+1}$,

$$\begin{aligned} f(\widetilde{A_{p,p+1}}) &= f([\widetilde{I_{pp}}, \widetilde{A_{p,p+1}}]) \\ &= [f(\widetilde{I_{pp}}), \widetilde{A_{p,p+1}}] + [\widetilde{I_{pp}}, f(\widetilde{A_{p,p+1}})] \\ &= \sum_{i=1}^{p-1} \left\{ f(\widetilde{I_{pp}})_{ip} A_{p,p+1} \right\}_{i,p+1} + \sum_{j=p+1}^t \left\{ f(\widetilde{A_{p,p+1}})_{pj} \right\}_{pj} - \sum_{i=1}^{p-1} \left\{ f(\widetilde{A_{p,p+1}})_{ip} \right\}_{ip} \end{aligned}$$

where the last equality is given by Lemma 4.4. Therefore,

$$f(\widetilde{A_{p,p+1}}) + \sum_{i=1}^{p-1} \left\{ f(\widetilde{A_{p,p+1}})_{ip} \right\}_{ip} = \sum_{i=1}^{p-1} \left\{ f(\widetilde{I_{pp}})_{ip} A_{p,p+1} \right\}_{i,p+1} + \sum_{j=p+1}^t \left\{ f(\widetilde{A_{p,p+1}})_{pj} \right\}_{pj}$$

One one hand, as $\text{char}(\mathbb{F}) \neq 2$, the nonzero blocks on the left side of the above equality are those of $f(\widetilde{A_{p,p+1}})$; on the other hand, the right side of this equality has nonzero (i, j) blocks only for $1 \leq i \leq p - 1 < p + 1 = j$ or $i = p < p + 1 \leq j \leq t$. So $k = 1$ is done.

(b) $k = \ell$: Suppose the statement is true for all $k < \ell$ where $\ell \geq 2$. Now $\widetilde{M_{p,p+\ell}} = [\widetilde{M_{p,p+1}}, \widetilde{M_{p+1,p+\ell}}]$, and

$$f([\widetilde{B_{p,p+1}}, \widetilde{C_{p+1,p+\ell}}]) = [f(\widetilde{B_{p,p+1}}), \widetilde{C_{p+1,p+\ell}}] + [\widetilde{B_{p,p+1}}, f(\widetilde{C_{p+1,p+\ell}})]$$

By induction hypothesis, $f(\widetilde{B_{p,p+1}})$ has nonzero blocks only on the p block row and the $(p + 1)$ block column, so that the nonzero blocks of $[f(\widetilde{B_{p,p+1}}), \widetilde{C_{p+1,p+\ell}}]$ only locate on the p block row and the $(p + \ell)$ block row. Similarly for $[\widetilde{B_{p,p+1}}, f(\widetilde{C_{p+1,p+\ell}})]$. So (4.6) is true for $k = \ell$.

(c) Overall, (4.6) is verified for all the cases. \square

Now we are ready to prove Theorem 3.1. The basic idea is to explore what remain in $\text{Der}(M_{\mathcal{L}})$ after factoring out $\text{ad}(N(M_{\mathcal{L}}))|_{M_{\mathcal{L}}} = \text{ad}(M_{\mathcal{L}_B})|_{M_{\mathcal{L}}}$. Given $X \in M_{\mathcal{L}_B}$, $A \in M_{\mathcal{L}}$,

$$\text{ad } X(A) = \sum_{1 \leq p \leq q \leq t} \sum_{(i,j) \in [I(\mathcal{L})]} [\widetilde{X_{pq}}, \widetilde{A_{ij}}].$$

A summand $[\widetilde{X_{pq}}, \widetilde{A_{ij}}]$ is nonzero only if $i = q$ or $p = j$. In other words, $\text{ad } \widetilde{X_{pq}}$ has nonzero action only on the q block row or the p block column of A . It motivates us to investigate the relationship of $f(\widetilde{A_{ip}})$ and $f(\widetilde{A_{qj}})$ for given $f \in \text{Der}(M_{\mathcal{L}})$ and $1 \leq p \leq q \leq t$.

Proof of Theorem 3.1.

- (1) If $M_{\mathcal{L}}$ is a set of block upper triangular matrices (i.e. $\mathcal{L} = \mathcal{L}_B$), by [8, Theorem 4.1] and the assumption $\text{char}(\mathbb{F}) \neq 2$, every $f \in \text{Der}(M_{\mathcal{L}})$ corresponds to $X \in M_{\mathcal{L}}$ and $\mu \in M_{\mathcal{L}}^*$ such that

$$f(A) = \text{ad } X(A) + \mu(A)I_n.$$

Then $\mu([M_{\mathcal{L}}, M_{\mathcal{L}}]) = 0$ by derivation property. All $\widetilde{\mathcal{M}}_{ij}$ with $i < j$ are in $[M_{\mathcal{L}}, M_{\mathcal{L}}]$. So $\mu(A) = \sum_{k \in [t]} \mu(\widetilde{A_{kk}})$. Recall that the (p, q) standard matrix in \mathcal{M}_{ij} is denoted by $E_{pq}^{[ij]}$. Given $k \in [t]$, we have $\widetilde{A_{kk}} - \text{tr}(A_{kk})\widetilde{E_{11}^{[kk]}} \in [M_{\mathcal{L}}, M_{\mathcal{L}}]$ so that

$$\mu(\widetilde{A_{kk}}) = \text{tr}(A_{kk})\mu(\widetilde{E_{11}^{[kk]}}).$$

Denote $c_k = \mu(\widetilde{E_{11}^{[kk]}})$. Then

$$f(A) = \text{ad } X(A) + \left(\sum_{k \in [t]} c_k \text{tr}(A_{kk}) \right) I_n.$$

This is (3.4). The formulae (3.1) and (3.2) for $\mathcal{L} = \mathcal{L}_B$ immediately follow.

- (2) *In the remaining of the proof, we assume $\mathcal{L} \neq \mathcal{L}_B$, so that $M_{\mathcal{L}}$ has at least one zero diagonal block with respect to the partition $\gamma_{\mathcal{L}}$.*

Suppose $(k, k) \in [I(\mathcal{L})]$. For any $A_{kk}, B_{kk} \in \mathcal{M}_{kk}$,

$$f(\widetilde{A_{kk}}, \widetilde{B_{kk}})_{kk} = [f(\widetilde{A_{kk}})_{kk}, B_{kk}] + [A_{kk}, f(\widetilde{B_{kk}})_{kk}].$$

So $f(\widetilde{\cdot})_{kk} : \mathcal{M}_{kk} \rightarrow \mathcal{M}_{kk}$ is a derivation of \mathcal{M}_{kk} . Since $\text{char}(\mathbb{F}) \neq 2$, according to [8, Corollary 5.1]¹, there is $X_{kk} \in \mathcal{M}_{kk}$ and $\lambda_k \in \mathbb{F}$ such that

$$f(\widetilde{A_{kk}})_{kk} = [X_{kk}, A_{kk}] + \lambda_k \text{tr}(A_{kk})I_{kk} \quad \text{for } A_{kk} \in \mathcal{M}_{kk}.$$

We prove that $\lambda_k = 0$ for all k . Recall that $E_{pq}^{[ij]}$ denotes the (p, q) standard matrix in \mathcal{M}_{ij} . On one hand, the $(1, 1)$ entry of

$$f(\widetilde{E_{11}^{[kk]}})_{kk} = [X_{kk}, E_{11}^{[kk]}] + \lambda_k I_{kk}$$

equals to λ_k . On the other hand, for any $\ell \in [t]$ with $\ell > k$,

$$\begin{aligned} f(\widetilde{E_{11}^{[k\ell]}})_{k\ell} &= f(\widetilde{[E_{11}^{[kk]}, E_{11}^{[k\ell]}]})_{k\ell} = [f(\widetilde{E_{11}^{[kk]}}), \widetilde{E_{11}^{[k\ell]}}]_{k\ell} + [\widetilde{E_{11}^{[kk]}}, f(\widetilde{E_{11}^{[k\ell]}})]_{k\ell} \\ &= f(\widetilde{E_{11}^{[kk]}})_{kk} E_{11}^{[k\ell]} - E_{11}^{[k\ell]} f(\widetilde{E_{11}^{[kk]}})_{\ell\ell} + E_{11}^{[kk]} f(\widetilde{E_{11}^{[k\ell]}})_{k\ell}. \end{aligned}$$

Therefore,

$$f(\widetilde{E_{11}^{[kk]}})_{kk} E_{11}^{[k\ell]} = (I_{kk} - E_{11}^{[kk]}) f(\widetilde{E_{11}^{[k\ell]}})_{k\ell} + E_{11}^{[k\ell]} f(\widetilde{E_{11}^{[kk]}})_{\ell\ell}.$$

Comparing the $(1, 1)$ entry of both sides, we see that the $(1, 1)$ entries of $f(\widetilde{E_{11}^{[kk]}})_{kk}$ and $f(\widetilde{E_{11}^{[k\ell]}})_{\ell\ell}$ are equal. The same result holds for $\ell < k$. By assumption $\mathcal{L} \neq \mathcal{L}_B$.

¹ $\text{Der}(\mathfrak{gl}(m, \mathbb{F}))$ has additional elements when $\text{char}(\mathbb{F}) = 2$ and $m = 2$ [8, Corollary 5.1].

So there exists $(\ell, \ell) \notin [I(\mathcal{L})]$, where $f(\widetilde{E_{11}^{[kk]}})_{\ell\ell} = 0$. Hence $\lambda_k = 0$. Overall, for any $(k, k) \in [I(\mathcal{L})]$, there exists $X_{kk} \in \mathcal{M}_{kk}$ such that

$$f(\widetilde{A_{kk}})_{kk} = [X_{kk}, A_{kk}] \quad \text{for all } A_{kk} \in \mathcal{M}_{kk}.$$

(3) Given $p, q \in [t]$ and $p < q$, we claim that there exists $X_{pq} \in \mathcal{M}_{pq}$ such that

$$f(\widetilde{A_{ip}})_{iq} = \text{ad } \widetilde{X_{pq}}(\widetilde{A_{ip}}), \quad \text{for any } (i, p) \in [I(\mathcal{L})], \quad \text{and} \quad (4.8)$$

$$f(\widetilde{A_{qj}})_{pj} = \text{ad } \widetilde{X_{pq}}(\widetilde{A_{qj}}), \quad \text{for any } (q, j) \in [I(\mathcal{L})]. \quad (4.9)$$

There are several situations:

(a) Suppose $(q, j) = (t, t) \in [I(\mathcal{L})]$. For any $A_{tt}, B_{tt} \in \mathcal{M}_{tt}$,

$$f([\widetilde{A_{tt}}, \widetilde{B_{tt}}])_{pt} = [f(\widetilde{A_{tt}}), \widetilde{B_{tt}}]_{pt} + [\widetilde{A_{tt}}, f(\widetilde{B_{tt}})]_{pt} = f(\widetilde{A_{tt}})_{pt} B_{tt} - f(\widetilde{B_{tt}})_{pt} A_{tt}.$$

Set $B_{tt} = I_{tt}$. Then $f(\widetilde{A_{tt}})_{pt} = f(\widetilde{I_{tt}})_{pt} A_{tt}$ for $A_{tt} \in \mathcal{M}_{tt}$. Denote $X_{pt} := f(\widetilde{I_{tt}})_{pt} \in \mathcal{M}_{pt}$. We have $f(\widetilde{A_{tt}})_{pt} = X_{pt} A_{tt}$ and so

$$f(\widetilde{A_{tt}})_{pt} = \text{ad } \widetilde{X_{pt}}(\widetilde{A_{tt}}) \quad \text{for all } A_{tt} \in \mathcal{M}_{tt}.$$

(b) Suppose $(i, p) = (1, 1) \in [I(\mathcal{L})]$. Similarly, let $Y_{1q} := -f(\widetilde{I_{11}})_{1q} \in \mathcal{M}_{1q}$ then

$$f(\widetilde{A_{11}})_{1q} = -\widetilde{A_{11}} Y_{1q} = \text{ad } \widetilde{Y_{1q}}(\widetilde{A_{11}}) \quad \text{for all } A_{11} \in \mathcal{M}_{11}.$$

(c) Suppose $(q, j) \in [I(\mathcal{L})] - \{(t, t)\}$. Either $q < t$ or $j < t$. Without loss of generality, suppose $j < t$. Let $j' := j + 1$. Then $(j, j'), (q, j'), (p, j), (p, j') \in [I(\mathcal{L})]$, and $\widetilde{\mathcal{M}_{qj'}} = \widetilde{\mathcal{M}_{qj}} \widetilde{\mathcal{M}_{jj'}} = [\widetilde{\mathcal{M}_{qj}}, \widetilde{\mathcal{M}_{jj'}}]$. For any $A_{qj} \in \mathcal{M}_{qj}$, $A_{jj'} \in \mathcal{M}_{jj'}$,

$$f(\widetilde{A_{qj} A_{jj'}})_{pj'} = f([\widetilde{A_{qj}}, \widetilde{A_{jj'}}])_{pj'} = [f(\widetilde{A_{qj}}), \widetilde{A_{jj'}}]_{pj'} + [\widetilde{A_{qj}}, f(\widetilde{A_{jj'}})]_{pj'} = f(\widetilde{A_{qj}})_{pj} A_{jj'}.$$

Applying Lemma 4.2 to $\phi : \mathcal{M}_{qj'} \rightarrow \mathcal{M}_{pj'}$ defined by $\phi(C) := f(\widetilde{C})_{pj'}$ and $\varphi : \mathcal{M}_{qj} \rightarrow \mathcal{M}_{pj}$ defined by $\varphi(D) := f(\widetilde{D})_{pj}$, we can find $X_{pq} \in \mathcal{M}_{pq}$ such that $f(\widetilde{A_{qj}})_{pj} = X_{pq} A_{qj}$ for all $A_{qj} \in \mathcal{M}_{qj}$, and $f(\widetilde{A_{qj'}})_{pj'} = X_{pq} A_{qj'}$ for all $A_{qj'} \in \mathcal{M}_{qj'}$. In particular, X_{pq} is independent of j . So

$$f(\widetilde{A_{qj}})_{pj} = \text{ad } \widetilde{X_{pq}}(\widetilde{A_{qj}}) \quad \text{for all } A_{qj} \in \mathcal{M}_{qj}.$$

(d) Suppose $(i, p) \in [I(\mathcal{L})] - \{(1, 1)\}$. Either $i > 1$ or $p > 1$. Without loss of generality, suppose $i > 1$ (similarly for $p > 1$). Let $i' := i - 1$. Then $(i', i), (i', p), (i, q), (i', q) \in [I(\mathcal{L})]$, and $\widetilde{\mathcal{M}_{i'p}} = \widetilde{\mathcal{M}_{i'i}} \widetilde{\mathcal{M}_{ip}} = [\widetilde{\mathcal{M}_{i'i}}, \widetilde{\mathcal{M}_{ip}}]$. For $A_{i'i} \in \mathcal{M}_{i'i}$ and $A_{ip} \in \mathcal{M}_{ip}$,

$$f(\widetilde{A_{i'i} A_{ip}})_{i'q} = f([\widetilde{A_{i'i}}, \widetilde{A_{ip}}])_{i'q} = [f(\widetilde{A_{i'i}}), \widetilde{A_{ip}}]_{i'q} + [\widetilde{A_{i'i}}, f(\widetilde{A_{ip}})]_{i'q} = A_{i'i} f(\widetilde{A_{ip}})_{i'q}.$$

Applying Lemma 4.1 to $\phi : \mathcal{M}_{i'p} \rightarrow \mathcal{M}_{i'q}$ defined by $\phi(C) := f(\widetilde{C})_{i'q}$ and $\varphi : \mathcal{M}_{ip} \rightarrow \mathcal{M}_{iq}$ defined by $\varphi(D) := f(\widetilde{D})_{iq}$, we can find $-Y_{pq} \in \mathcal{M}_{pq}$ such that $f(\widetilde{A_{ip}})_{iq} = -A_{ip} Y_{pq}$ for all $A_{ip} \in \mathcal{M}_{ip}$, and $f(\widetilde{A_{i'p}})_{i'q} = -A_{i'p} Y_{pq}$ for all $A_{i'p} \in \mathcal{M}_{i'p}$. So $-Y_{pq}$ is independent of i and

$$f(\widetilde{A_{ip}})_{iq} = \text{ad } \widetilde{Y_{pq}}(\widetilde{A_{ip}}), \quad \text{for all } A_{ip} \in \mathcal{M}_{ip}.$$

(e) Given any $(i, p), (q, j) \in [I(\mathcal{L})]$, we have $[\widetilde{A}_{ip}, \widetilde{A}_{qj}] = 0$, so that

$$\begin{aligned} 0 &= f([\widetilde{A}_{ip}, \widetilde{A}_{qj}])_{ij} = [f(\widetilde{A}_{ip}), \widetilde{A}_{qj}]_{ij} + [\widetilde{A}_{ip}, f(\widetilde{A}_{qj})]_{ij} \\ &= f(\widetilde{A}_{ip})_{iq} A_{qj} + A_{ip} f(\widetilde{A}_{qj})_{pj} = -A_{ip} Y_{pq} A_{qj} + A_{ip} X_{pq} A_{qj}. \end{aligned}$$

Therefore, $X_{pq} = Y_{pq}$.

Overall, we successfully find X_{pq} that satisfies (4.8) and (4.9).

(4) From (2) and (3), we can construct a matrix in $\mathcal{M}_{\mathcal{L}}$:

$$X_0 := \sum_{(k,k) \in [I(\mathcal{L})]} \widetilde{X}_{kk} + \sum_{1 \leq p < q \leq t} \widetilde{X}_{pq}.$$

Define the derivation

$$f_1 := f - \text{ad } X_0. \quad (4.10)$$

Then for any $(k, k) \in [I(\mathcal{L})]$, $1 \leq p < q \leq t$, and $(i, p), (q, j) \in [I(\mathcal{L})]$, we have

$$f_1(\widetilde{\mathcal{M}}_{kk})_{kk} = 0, \quad f_1(\widetilde{\mathcal{M}}_{ip})_{iq} = 0, \quad f_1(\widetilde{\mathcal{M}}_{qj})_{pj} = 0.$$

By Lemmas 4.4 and 4.5, f_1 belongs to the following set:

$$\begin{aligned} D_0 := \{g \in \text{Der}(M_{\mathcal{L}}) \mid & g(\widetilde{\mathcal{M}}_{kk}) \in Z(M_{\mathcal{L}}) \cap M_{\mathcal{L}} \text{ for } (k, k) \in [I(\mathcal{L})], \\ & g(\widetilde{\mathcal{M}}_{pq}) \subseteq \widetilde{\mathcal{M}}_{pq} \text{ for } 1 \leq p < q \leq t\}. \end{aligned} \quad (4.11)$$

It remains to describe the subalgebra D_0 of $\text{Der}(M_{\mathcal{L}})$.

(5) Given $f' \in \text{Der}(M_{\mathcal{L}})$, $p, q \in [t]$ with $p < q$, and $k \in [t]$ with $p \leq k \leq q$, Lemmas 4.4 and 4.5 imply that

$$\begin{aligned} f'(\widetilde{A_{pk} A_{kq}})_{pq} &= f'([\widetilde{A_{pk}}, \widetilde{A_{kq}}])_{pq} = [f'(\widetilde{A_{pk}}), \widetilde{A_{kq}}]_{pq} + [\widetilde{A_{pk}}, f'(\widetilde{A_{kq}})]_{pq} \\ &= f'(\widetilde{A_{pk}})_{pk} A_{kq} + A_{pk} f'(\widetilde{A_{kq}})_{kq}. \end{aligned} \quad (4.12)$$

This formula will be frequently used in the following computations.

(6) We prove the following claim regarding f_1 defined in (4.10): there exist $Y_{ii} \in \mathcal{M}_{ii}$ for $i \in [t]$, such that for each $k \in [t]$, the derivation $f_1^{(k)} := \left(f_1 - \sum_{i=1}^k \text{ad } \widetilde{Y}_{ii}\right)\Big|_{M_{\mathcal{L}}}$ has the images

$$\begin{cases} f_1^{(k)}(\widetilde{\mathcal{M}}_{qq}) = f_1(\widetilde{\mathcal{M}}_{qq}), & \text{for } (q, q) \in [I(\mathcal{L})], \ q \leq k; \\ f_1^{(k)}(\widetilde{\mathcal{M}}_{pq}) = 0, & \text{for } (p, q) \in [I(\mathcal{L})], \ 1 \leq p < q \leq k. \end{cases} \quad (4.13)$$

Moreover, $Y_{ii} \in \mathbb{F}I_{ii}$ whenever $(i, i) \in [I(\mathcal{L})]$.

The proof is proceeded by induction on k :

(a) $k = 1$ and 2: There are two subcases:

• If $(1, 1) \in [I(\mathcal{L})]$, we let $Y_{11} = 0 \in \mathcal{M}_{11}$ so that $f_1^{(1)} = f_1$. By (4.12),

$$f_1^{(1)}(\widetilde{A_{11} A_{12}})_{12} = f_1^{(1)}(\widetilde{A_{11}})_{11} A_{12} + A_{11} f_1^{(1)}(\widetilde{A_{12}})_{12} = A_{11} f_1^{(1)}(\widetilde{A_{12}})_{12}.$$

By Lemma 4.1, there exists $-Y_{22} \in \mathcal{M}_{22}$, such that $f_1^{(1)}(\widetilde{A_{12}})_{12} = -A_{12} Y_{22}$.

Let $f_1^{(2)} = f_1^{(1)} - \text{ad } \widetilde{Y_{22}}$. Then $f_1^{(2)}(\widetilde{A_{12}}) = 0$. If furthermore $(2, 2) \in [I(\mathcal{L})]$, then by (4.12),

$$0 = f_1^{(2)}(\widetilde{A_{12} A_{22}})_{12} = f_1^{(2)}(\widetilde{A_{12}})_{12} A_{22} + A_{12} f_1^{(2)}(\widetilde{A_{22}})_{22} = A_{12} f_1^{(2)}(\widetilde{A_{22}})_{22}.$$

Thus

$$0 = f_1^{(2)}(\widetilde{A_{22}})_{22} = f_1(\widetilde{A_{22}})_{22} - [Y_{22}, A_{22}] = -[Y_{22}, A_{22}].$$

So $Y_{22} \in \mathbb{F}I_{22}$ and $f_1^{(2)}(\widetilde{A_{22}}) = f_1(\widetilde{A_{22}})$. The claim holds for $k = 1, 2$.

- If $(1, 1) \notin [I(\mathcal{L})]$, then $(2, 2) \in [I(\mathcal{L})]$ by Theorem 2.4. By (4.12),

$$f_1(\widetilde{A_{12}A_{22}})_{12} = f_1(\widetilde{A_{12}})_{12}A_{22} + A_{12}f_1(\widetilde{A_{22}})_{22} = f_1(\widetilde{A_{12}})_{12}A_{22}.$$

By Lemma 4.2, there exists $Y_{11} \in \mathcal{M}_{11}$ such that $f_1(\widetilde{A_{12}})_{12} = Y_{11}A_{12}$. Let $Y_{22} = 0 \in \mathcal{M}_{22}$, $f_1^{(1)} = f_1 - \text{ad } \widetilde{Y_{11}}$, and $f_1^{(2)} = f_1^{(1)} - \text{ad } \widetilde{Y_{22}}$. Then the claim holds for $k = 1, 2$.

- (b) $k = \ell > 2$: Suppose the claim holds for $k = \ell - 1 \geq 2$. So there exist $Y_{11} \in \mathcal{M}_{11}, \dots, Y_{\ell-1, \ell-1} \in \mathcal{M}_{\ell-1, \ell-1}$, such that $f_1^{(\ell-1)} := f_1 - \sum_{i=1}^{\ell-1} \text{ad } \widetilde{Y_{ii}}$ satisfies (4.13) for $k = \ell - 1$. Clearly $f_1^{(\ell-1)} \in D_0$. For any $p \in [\ell - 2]$, by (4.12),

$$\begin{aligned} f_1^{(\ell-1)}(\widetilde{A_{p, \ell-1}A_{\ell-1, \ell}})_{p\ell} &= f_1^{(\ell-1)}(\widetilde{A_{p, \ell-1}})_{p, \ell-1}A_{\ell-1, \ell} + A_{p, \ell-1}f_1^{(\ell-1)}(\widetilde{A_{\ell-1, \ell}})_{\ell-1, \ell} \\ &= A_{p, \ell-1}f_1^{(\ell-1)}(\widetilde{A_{\ell-1, \ell}})_{\ell-1, \ell}. \end{aligned}$$

By Lemma 4.1, there exists $-Y_{\ell\ell} \in \mathcal{M}_{\ell\ell}$, such that

$$f_1^{(\ell-1)}(\widetilde{A_{p\ell}})_{p\ell} = -A_{p\ell}Y_{\ell\ell} \quad \text{for all } p \in [\ell - 1].$$

Let

$$f_1^{(\ell)} := f_1^{(\ell-1)} - \text{ad } \widetilde{Y_{\ell\ell}}.$$

Then $f_1^{(\ell)}(\widetilde{A_{p\ell}}) = 0$ for $p \in [\ell - 1]$. In the case $(\ell, \ell) \in [I(\mathcal{L})]$, by (4.12),

$$0 = f_1^{(\ell)}(\widetilde{A_{\ell-1, \ell}A_{\ell\ell}})_{\ell-1, \ell} = f_1^{(\ell)}(\widetilde{A_{\ell-1, \ell}})_{\ell-1, \ell}A_{\ell\ell} + A_{\ell-1, \ell}f_1^{(\ell)}(\widetilde{A_{\ell\ell}})_{\ell\ell} = A_{\ell-1, \ell}f_1^{(\ell)}(\widetilde{A_{\ell\ell}})_{\ell\ell}.$$

So

$$0 = f_1^{(\ell)}(\widetilde{A_{\ell\ell}})_{\ell\ell} = \left(f_1 - \sum_{i=1}^{\ell} \text{ad } \widetilde{Y_{ii}} \right) (\widetilde{A_{\ell\ell}})_{\ell\ell} = -[Y_{\ell\ell}, A_{\ell\ell}].$$

Thus $Y_{\ell\ell} \in \mathbb{F}I_{\ell\ell}$ and $f_1^{(\ell)}(\widetilde{A_{\ell\ell}}) = f_1(\widetilde{A_{\ell\ell}})$. The claim is proved for $k = \ell$.

- (c) Overall, the claim holds for every $k \in [t]$.

- (7) The derivation $f_1^{(t)} = f_1 - \sum_{i=1}^t \text{ad } \widetilde{Y_{ii}}$ sends each $\widetilde{\mathcal{M}_{kk}}$ for $(k, k) \in [I(\mathcal{L})]$ to $Z(M_{\mathcal{L}}) \cap M_{\mathcal{L}}$, and $\widetilde{\mathcal{M}_{pq}}$ for $1 \leq p < q \leq t$ to 0. For any $A, B \in M_{\mathcal{L}}$,

$$f_1^{(t)}([A, B]) = [f_1^{(t)}(A), B] + [A, f_1^{(t)}(B)] = 0.$$

Therefore, $f_1^{(t)} \in \mathcal{D}$ for \mathcal{D} defined in (3.3). Every $\phi \in \mathcal{D}$ satisfies $\phi([A, B]) = 0 = [\phi(A), B] + [A, \phi(B)]$ for $A, B \in M_{\mathcal{L}}$. Thus $\mathcal{D} \subseteq \text{Der } M_{\mathcal{L}}$. So far we have

$$\text{Der}(M_{\mathcal{L}}) = (\text{ad } M_{\mathcal{L}_B})|_{M_{\mathcal{L}}} + \mathcal{D}.$$

If $(1, 1) \in [I(\mathcal{L})]$ or $(t, t) \in [I(\mathcal{L})]$, then $Z(M_{\mathcal{L}}) \cap M_{\mathcal{L}} = 0$ implies that $\mathcal{D} = 0$. We get (3.5).

If neither $(1, 1)$ nor (t, t) is in $[I(\mathcal{L})]$, then $Z(M_{\mathcal{L}}) \cap M_{\mathcal{L}} = \widetilde{\mathcal{M}_{1t}}$. The set $\{\widetilde{E_{11}^{[kk]}} \mid (k, k) \in [I(\mathcal{L})]\}$ spans a subalgebra complement to $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ in $M_{\mathcal{L}}$. Then for any

The above observations indicate that the structure of $\text{Der}(M_{\mathcal{L}}^0)$ for SDUT ladders \mathcal{L} (where $\mathcal{L} = \mathcal{L}_*$) will be useful in studying the structure of $\text{Der}(M_{\mathcal{L}'})$ for non-solvable upper triangular ladders \mathcal{L}' . In the rest of this section, we assume that \mathcal{L} is a SDUT ladder, unless otherwise specified. Let $t := |\gamma_{\mathcal{L}}| + 1$ as before.

Theorem 5.3. *Suppose $\text{char}(\mathbb{F}) \neq 2, 3$. Let \mathcal{L} be a SDUT ladder of size n . Then every derivation $f \in \text{Der}(M_{\mathcal{L}}^0)$ can be extended to a derivation $f^+ \in \text{Der}(M_{\mathcal{L}})$ such that $f^+|_{M_{\mathcal{L}}^0} = f$. In particular, there exists a block upper triangular matrix $X \in M_{\mathcal{L}_B}$ such that*

$$f(B) = \text{ad } X(B) = [X, B], \quad \text{for all } B \in M_{\mathcal{L}}^0. \quad (5.1)$$

We can write

$$\text{Der}(M_{\mathcal{L}}^0) = \text{ad}(\text{N}(M_{\mathcal{L}})/\text{Z}(M_{\mathcal{L}}))|_{M_{\mathcal{L}}^0}. \quad (5.2)$$

The proof of Theorem 5.3 will be deferred to the end of this section.

Corollary 5.4. *When $\text{char}(\mathbb{F}) \neq 2, 3$, and \mathcal{L} is a SDUT ladder, we have the split exact sequence:*

$$0 \hookrightarrow \mathcal{D} \hookrightarrow \text{Der}(M_{\mathcal{L}}) \xrightarrow{\pi} \text{Der}(M_{\mathcal{L}}^0) \rightarrow 0, \quad (5.3)$$

where \mathcal{D} is defined in (3.3).

Proof. Theorem 5.3 shows that the restriction map $\pi : \text{Der}(M_{\mathcal{L}}) \rightarrow \text{Der}(M_{\mathcal{L}}^0)$ is surjective. Theorem 3.1 shows that $\text{Der}(M_{\mathcal{L}}) = \text{ad}(\text{N}(M_{\mathcal{L}})/\text{Z}(M_{\mathcal{L}}))|_{M_{\mathcal{L}}} \oplus \mathcal{D}$. It is easy to check that $\text{Z}(M_{\mathcal{L}}) = \text{Z}(M_{\mathcal{L}}^0)$ and $\text{Ker } \pi = \mathcal{D}$. Therefore, we get the split exact sequence (5.3). \square

Example 5.5. *When $\text{char}(\mathbb{F}) = 2$ or 3 , we show by counterexamples that $\text{Der}(M_{\mathcal{L}}^0)$ is not in the form of (5.2).*

- $\text{char}(\mathbb{F}) = 2$: Let $M_{\mathcal{L}} = M_2$, so that $M_{\mathcal{L}}^0 = \mathfrak{sl}_2$. Let f be the derivation of M_2 given in Example 3.2, that is, $f(E_{12}) = E_{21}$, and $f(E_{ij}) = 0$ for $(i, j) \in \{(1, 1), (2, 2), (2, 1)\}$. Then $f|_{\mathfrak{sl}_2}$ is a derivation of \mathfrak{sl}_2 . However, there is no $X \in M_{\mathcal{L}_B} = M_2$ such that $f|_{\mathfrak{sl}_2}(E_{12}) = [X, E_{12}]$.
- $\text{char}(\mathbb{F}) = 3$: Let $n = 4$, $\mathcal{L} = \{(2, 1)\}$. Then $M_{\mathcal{L}}^0$ consists of matrices in M_4 that takes the following forms:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & -a_{11} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a_{ij} \in \mathbb{F}.$$

So $M_{\mathcal{L}}^0$ has a basis $\mathcal{B} = \{E_{11} - E_{22}, E_{12}, E_{13}, E_{14}, E_{21}, E_{23}, E_{24}\}$. Define $f \in \text{End}(M_{\mathcal{L}}^0)$ by $f(E_{12}) := E_{24}$, and $f(E) = 0$ for all other matrices E in the basis \mathcal{B} . We prove that

$$f([E, E']) = [f(E), E'] + [E, f(E')] \quad (5.4)$$

for any distinct $E, E' \in \mathcal{B}$, so that $f \in \text{Der}(M_{\mathcal{L}}^0)$. The only case that the left side or the right side of (5.4) is nonzero is $\{E, E'\} = \{E_{11} - E_{22}, E_{12}\}$, in which

$$f([E, E']) = 2f(E_{12}) = 2E_{24}, \quad [f(E), E'] + [E, f(E')] = -E_{24}.$$

Since $\text{char}(\mathbb{F}) = 3$, the equality (5.4) holds for this case. Therefore, (5.4) holds for all $\{E, E'\} \subseteq \mathcal{B}$, and $f \in \text{Der}(M_{\mathcal{L}}^0)$. However, there is no matrix $X \in M_4$, such that $f(E_{12}) = [X, E_{12}]$.

In order to prove Theorem 5.3, we first give two lemmas similar to Lemmas 4.1 and 4.2.

Lemma 5.6. *Suppose $n \geq 2$. If linear transformations $\phi : M_{mn} \rightarrow M_{mq}$ and $\varphi : \mathfrak{sl}_n \rightarrow M_{nq}$ satisfy that*

$$\phi(AB) = A\varphi(B) \quad \text{for all } A \in M_{mn}, B \in \mathfrak{sl}_n, \quad (5.5)$$

then there is $X \in M_{nq}$ such that $\phi(C) = CX$ for $C \in M_{mn}$ and $\varphi(D) = DX$ for $D \in \mathfrak{sl}_n$.

Lemma 5.6 is very similar to a special case ($p = n$) of Lemma 4.1, except that the domain of φ is \mathfrak{sl}_n instead of $M_{nn} = M_n$. The proof of Lemma 5.6 (omitted) is totally parallel to that of Lemma 4.1, using the key fact that $\{E_{1j}^{(mn)} B \mid j \in [n], B \in \mathfrak{sl}_n\}$ still spans the first row space of M_{mn} . Similarly, we have the following lemma.

Lemma 5.7. *Suppose $n \geq 2$. If linear transformations $\phi : M_{nq} \rightarrow M_{mq}$ and $\varphi : \mathfrak{sl}_n \rightarrow M_{mn}$ satisfy that*

$$\phi(BA) = \varphi(B)A \quad \text{for all } A \in M_{nq}, B \in \mathfrak{sl}_n, \quad (5.6)$$

then there is $X \in M_{mn}$ such that $\phi(C) = XC$ for $C \in M_{nq}$ and $\varphi(D) = XD$ for $D \in \mathfrak{sl}_n$.

Next we give two lemmas related to the bracket operation.

Lemma 5.8. *Suppose $\text{char}(\mathbb{F}) \neq 2, 3$. If a linear transformation $\phi : \mathfrak{sl}_n \rightarrow M_{nm}$ satisfies that*

$$\phi(AB - BA) = A\phi(B) - B\phi(A), \quad \text{for all } A, B \in \mathfrak{sl}_n, \quad (5.7)$$

then there is $X \in M_{nm}$ such that $\phi(C) = CX$ for $C \in \mathfrak{sl}_n$.

Proof. The case $n = 1$ is obviously true. We now assume that $n \geq 2$. Let $\{E_{ij} \mid i, j \in [n]\}$ be the standard basis of M_n . Then \mathfrak{sl}_n has the standard basis $\{E_{ij} \mid i, j \in [n], i \neq j\} \cup \{H_i \mid i \in [n-1]\}$, where $H_i := E_{ii} - E_{i+1, i+1}$. We have $M_n = \mathfrak{sl}_n \oplus \mathbb{F}E_{11}$.

First we prove that the only possibly nonzero row of $\phi(E_{ij})$ ($i \neq j$) is the i -th row, and the only possibly nonzero rows of $\phi(H_i) = \phi(E_{ii} - E_{i+1, i+1})$ ($i \in [n-1]$) are the i -th and the $(i+1)$ -th rows.

Suppose $i, j \in [n]$ with $i < j$. Denote $E := E_{ij}$, $F := E_{ji}$, and $H := E_{ii} - E_{jj}$. Then $\{H, E, F\} \in \mathfrak{sl}_n$ is the standard triple of a \mathfrak{sl}_2 subalgebra. We have

$$2\phi(E) = \phi([H, E]) = H\phi(E) - E\phi(H) \implies (2I_n - H)\phi(E) = -E\phi(H).$$

When $\text{char}(\mathbb{F}) \neq 2, 3$, the matrix $2I_n - H = \text{diag}(2, 2, \dots, 1, \dots, 3, \dots, 2)$ is invertible and diagonal. The matrix $(2I_n - H)^{-1}$ is again diagonal with 1 as the i -th diagonal entry. So we have

$$\phi(E) = -(2I_n - H)^{-1}E_{ij}\phi(H) = -E_{ij}\phi(H).$$

In particular, $\phi(E_{ij}) = \phi(E)$ has zeros outside of the i -th row. Similar argument works for E_{ji} .

For $H_i = E_{ii} - E_{i+1, i+1}$, we have

$$\phi(H_i) = \phi([E_{i, i+1}, E_{i+1, i}]) = E_{i, i+1}\phi(E_{i+1, i}) - E_{i+1, i}\phi(E_{i, i+1}).$$

Therefore, $\phi(H_i)$ has zeros outside of the i -th and the $(i+1)$ -th rows.

Next we extend the map ϕ from the domain \mathfrak{sl}_n to the domain M_n such that property (5.7) still hold in M_n . Define the linear transformation $\phi^+ : M_n \rightarrow M_{nm}$ as follow:

$$\begin{cases} \phi^+(A) = \phi(A), & \text{for } A \in \mathfrak{sl}_n; \\ \phi^+(E_{11}) = E_{12}\phi(E_{21}). \end{cases}$$

Then ϕ^+ is an extension of ϕ from \mathfrak{sl}_n to M_n . To verify (5.7)-like property for ϕ^+ in M_n , it suffices to prove the following equality for all A in the standard basis of \mathfrak{sl}_n :

$$\phi^+(E_{11}A - AE_{11}) = E_{11}\phi^+(A) - A\phi^+(E_{11}) = E_{11}\phi(A) - AE_{12}\phi(E_{21}). \quad (5.8)$$

- (1) $A = E_{1j}$, $1 \neq j \in [n]$: the left side of (5.8) is $\phi^+(E_{1j}) = \phi(E_{1j})$. The right side of (5.8) is $E_{11}\phi(E_{1j})$. Both sides are clearly equal since $\phi(E_{1j})$ has zero entries outside of the first row.
- (2) $A = E_{i1}$, $1 \neq i \in [n]$: the proof is similar.
- (3) $A = E_{ij}$, $i, j \in [n] - \{1\}$, $i \neq j$: both sides of (5.8) are zero.
- (4) $A = H_1 = E_{11} - E_{22}$: the left side of (5.8) is zero. The right side of (5.8) is

$$E_{11}\phi(H_1) - H_1E_{12}\phi(E_{21}) = E_{11}\phi(H_1) - E_{12}\phi(E_{21}).$$

We have

$$-2\phi(E_{21}) = \phi([H_1, E_{21}]) = H_1\phi(E_{21}) - E_{21}\phi(H_1) = -\phi(E_{21}) - E_{21}\phi(H_1),$$

where the last equality holds since $\phi(E_{21})$ has zeros outside of the second row. Therefore, $\phi(E_{21}) = E_{21}\phi(H_1)$, and the right side of (5.8) is

$$E_{11}\phi(H_1) - E_{12}\phi(E_{21}) = E_{11}\phi(H_1) - E_{12}E_{21}\phi(H_1) = 0.$$

So both sides are equal.

- (5) $A = H_i$, $i \in [n-1] - \{1\}$: Both sides of (5.8) are clearly zero.

Overall, (5.8) is proved. We have

$$\phi^+(AB - BA) = A\phi^+(B) - B\phi^+(A), \quad \text{for all } A, B \in M_n. \quad (5.9)$$

Finally, let $B = I_n$ in (5.9), then

$$0 = A\phi^+(I_n) - I_n\phi^+(A) \Rightarrow \phi^+(A) = A\phi^+(I_n).$$

Setting $X := \phi^+(I_n)$, we get $\phi(A) = AX$ for all $A \in \mathfrak{sl}_n$. □

Similarly, we have the following result.

Lemma 5.9. *Suppose $\text{char}(\mathbb{F}) \neq 2, 3$. If a linear transformation $\phi : \mathfrak{sl}_n \rightarrow M_{mn}$ satisfies that*

$$\phi(AB - BA) = \phi(A)B - \phi(B)A, \quad \text{for all } A, B \in \mathfrak{sl}_n, \quad (5.10)$$

then there is $X \in M_{mn}$ such that $\phi(C) = XC$ for $C \in \mathfrak{sl}_n$.

The statements of Lemmas 5.8 and 5.9 also hold when $\text{char}(\mathbb{F}) = 2$, but the proofs should be adjusted slightly. We will not need the case $\text{char}(\mathbb{F}) = 2$ here. The following counterexample shows that Lemma 5.8 is not true when $\text{char}(\mathbb{F}) = 3$. Likewise for Lemma 5.9.

Example 5.10. *Suppose $\text{char}(\mathbb{F}) = 3$. In M_2 , let $H := E_{11} - E_{22}$, and $\phi : \mathfrak{sl}_2 \rightarrow M_2$ the linear map given by*

$$\phi(E_{12}) := E_{21}, \quad \phi(E_{21}) := 0, \quad \phi(H) := 0.$$

Then ϕ satisfies (5.7) since

$$\begin{aligned} \phi([H, E_{12}]) &= 2\phi(E_{12}) = 2E_{21} = -E_{21} = H\phi(E_{12}) - E_{12}\phi(H), \\ \phi([H, E_{21}]) &= -2\phi(E_{21}) = 0 = H\phi(E_{21}) - E_{21}\phi(H), \\ \phi([E_{12}, E_{21}]) &= \phi(H) = 0 = E_{12}\phi(E_{21}) - E_{21}\phi(E_{12}). \end{aligned}$$

However, there is no $X \in M_2$ such that $\phi(E_{12}) = E_{21} = E_{12}X$.

Lemma 5.11. *Suppose $\text{char}(\mathbb{F}) \neq 2$. Then for any $f \in \text{Der}(M_{\mathcal{L}}^0)$:*

$$f(\widetilde{\mathfrak{sl}_{kk}}) \subseteq \widetilde{\mathfrak{sl}_{kk}} + \sum_{i=1}^{k-1} \widetilde{\mathcal{M}_{ik}} + \sum_{j=k+1}^t \widetilde{\mathcal{M}_{kj}}, \quad \text{for } (k, k) \in [I(\mathcal{L})]; \quad (5.11)$$

$$f(\widetilde{\mathcal{M}_{pq}}) \subseteq \widetilde{\mathcal{M}_{pq}} + \sum_{i=1}^{p-1} \widetilde{\mathcal{M}_{iq}} + \sum_{j=q+1}^t \widetilde{\mathcal{M}_{pj}}, \quad \text{for } 1 \leq p < q \leq t. \quad (5.12)$$

The proof is similar to that of Lemma 4.5, with some slight adjustments.

Proof. Given $(k, k) \in [I(\mathcal{L})]$, we have $[\widetilde{\mathfrak{sl}_{kk}}, \widetilde{\mathfrak{sl}_{kk}}] = \widetilde{\mathfrak{sl}_{kk}}$ in $M_{\mathcal{L}}^0$. For $A_{kk}, B_{kk} \in \mathfrak{sl}_{kk}$,

$$f([\widetilde{A_{kk}}, \widetilde{B_{kk}}]) = [f(\widetilde{A_{kk}}), \widetilde{B_{kk}}] + [\widetilde{A_{kk}}, f(\widetilde{B_{kk}})] \in \widetilde{\mathfrak{sl}_{kk}} + \sum_{i=1}^{k-1} \widetilde{\mathcal{M}_{ik}} + \sum_{j=k+1}^t \widetilde{\mathcal{M}_{kj}}.$$

So (5.11) is done.

Given $1 \leq p < q \leq t$, we prove (5.12) by induction on $\ell := q - p$:

- (1) $\ell = 1$: Here $(p, q) = (p, p+1) \in [I(\mathcal{L})]$. By Theorem 2.4, at least one of (p, p) and $(p+1, p+1)$ is in $[I(\mathcal{L})]$. Without loss of generality, suppose $(p, p) \in [I(\mathcal{L})]$. Since \mathcal{L} is SDUT, the matrices in \mathfrak{sl}_{pp} have the size $m \geq 2$. Therefore $[\widetilde{\mathfrak{sl}_{pp}}, \widetilde{\mathcal{M}_{p,p+1}}] = \widetilde{\mathcal{M}_{p,p+1}}$ in $M_{\mathcal{L}}^0$. Let $\{\cdot\}_{ij}$ also denote the embedding of \mathcal{M}_{ij} to $\widetilde{\mathcal{M}_{ij}} \in M_n$. For $A_{pp} \in \mathfrak{sl}_{pp}$, $A_{p,p+1} \in \mathcal{M}_{p,p+1}$,

$$\begin{aligned} f(\widetilde{A_{pp}A_{p,p+1}}) &= f([\widetilde{A_{pp}}, \widetilde{A_{p,p+1}}]) = [f(\widetilde{A_{pp}}), \widetilde{A_{p,p+1}}] + [\widetilde{A_{pp}}, f(\widetilde{A_{p,p+1}})] \\ &\in \widetilde{\mathcal{M}_{p,p+1}} + \sum_{i=1}^{p-1} \widetilde{\mathcal{M}_{i,p+1}} + \sum_{j=p+2}^t \widetilde{\mathcal{M}_{pj}} \\ &\quad - \sum_{i=1}^{p-1} \left\{ f(\widetilde{A_{p,p+1}})_{ip} A_{pp} \right\}_{ip} + \left\{ [A_{pp}, f(\widetilde{A_{p,p+1}})]_{pp} \right\}_{pp}. \end{aligned} \quad (5.13)$$

To get (5.12) for $q - p = 1$, it remains to prove that $f(\widetilde{E_{kj}^{[p,p+1]}})_{ip} = 0$ for any given standard matrix $E_{kj}^{[p,p+1]}$ in $\mathcal{M}_{p,p+1}$ and $i \in [p]$. There are two cases:

- $i \in [p-1]$: (5.13) shows that for $A_{pp} \in \mathfrak{sl}_{pp}$ and $A_{p,p+1} \in \mathcal{M}_{p,p+1}$,

$$f(\widetilde{A_{pp}A_{p,p+1}})_{ip} = -f(\widetilde{A_{p,p+1}})_{ip}A_{pp}. \quad (5.14)$$

Since the size m of \mathfrak{sl}_{pp} is no less than 2, we can choose $s \in [m] - \{k\}$. Then

$$f(\widetilde{E_{kj}^{[p,p+1]}})_{ip} = f(\widetilde{E_{ks}^{[pp]}E_{sj}^{[p,p+1]}})_{ip} = -f(\widetilde{E_{sj}^{[p,p+1]}})_{ip}E_{ks}^{[pp]}. \quad (5.15)$$

However, we also have

$$\begin{aligned}
0 &= f(\widetilde{[E_{kj}^{[p,p+1]}]}, \widetilde{[E_{sj}^{[p,p+1]}]})_{i,p+1} \\
&= [f(\widetilde{[E_{kj}^{[p,p+1]}]}), \widetilde{[E_{sj}^{[p,p+1]}]})_{i,p+1} + [\widetilde{[E_{kj}^{[p,p+1]}]}, f(\widetilde{[E_{sj}^{[p,p+1]}]})]_{i,p+1} \\
&= f(\widetilde{[E_{kj}^{[p,p+1]}]})_{ip} E_{sj}^{[p,p+1]} - f(\widetilde{[E_{sj}^{[p,p+1]}]})_{ip} E_{kj}^{[p,p+1]} \\
&= -f(\widetilde{[E_{sj}^{[p,p+1]}]})_{ip} E_{ks}^{[pp]} E_{sj}^{[p,p+1]} - f(\widetilde{[E_{sj}^{[p,p+1]}]})_{ip} E_{kj}^{[p,p+1]} \quad (\text{by (5.15)}) \\
&= -2f(\widetilde{[E_{sj}^{[p,p+1]}]})_{ip} E_{kj}^{[p,p+1]}.
\end{aligned}$$

Since $\text{char}(\mathbb{F}) \neq 2$, the k -th column of $f(\widetilde{[E_{sj}^{[p,p+1]}]})_{ip}$ must be zero. Then (5.15)

shows that $f(\widetilde{[E_{kj}^{[p,p+1]}]})_{ip} = -f(\widetilde{[E_{sj}^{[p,p+1]}]})_{ip} E_{ks}^{[pp]} = 0$.

- $i = p$: (5.13) shows that for $A_{pp} \in \mathfrak{sl}_{pp}$ and $A_{p,p+1} \in \mathcal{M}_{p,p+1}$,

$$f(\widetilde{[A_{pp} A_{p,p+1}]})_{pp} = [A_{pp}, f(\widetilde{[A_{p,p+1}]})_{pp}] = A_{pp} f(\widetilde{[A_{p,p+1}]})_{pp} - f(\widetilde{[A_{p,p+1}]})_{pp} A_{pp}.$$

In particular, for $r \in [m] - \{k\}$, we have $E_{kr}^{[pp]} \in \mathfrak{sl}_{pp}$ and

$$f(\widetilde{[E_{kj}^{[p,p+1]}]})_{pp} = f(\widetilde{[E_{kr}^{[pp]} E_{rj}^{[p,p+1]}]})_{pp} = E_{kr}^{[pp]} f(\widetilde{[E_{rj}^{[p,p+1]}]})_{pp} - f(\widetilde{[E_{rj}^{[p,p+1]}]})_{pp} E_{kr}^{[pp]}. \quad (5.16)$$

Denote

$$A = [a_{ij}]_{m \times m} := f(\widetilde{[E_{kj}^{[p,p+1]}]})_{pp}.$$

(5.16) implies that all nonzero entries of A are located in the k -th row and the r -th column. If $m \geq 3$, we can replace r by any $s \in [m] - \{k, r\}$ in (5.16) to show that all nonzero entries of A are located in the k -th row. In both $m = 2$ and $m \geq 3$ cases, we have

$$A = E_{kk}^{[pp]} A + a_{rr} E_{rr}^{[pp]}. \quad (5.17)$$

Applying (5.16) twice, we get

$$\begin{aligned}
A &= \left[E_{kr}^{[pp]}, f(\widetilde{[E_{rj}^{[p,p+1]}]})_{pp} \right] = \left[E_{kr}^{[pp]}, \left[E_{rk}^{[pp]}, f(\widetilde{[E_{kj}^{[p,p+1]}]})_{pp} \right] \right] \\
&= E_{kk}^{[pp]} A - E_{kr}^{[pp]} A E_{rk}^{[pp]} - E_{rk}^{[pp]} A E_{kr}^{[pp]} + A E_{rr}^{[pp]} \\
&= (A - a_{rr} E_{rr}^{[pp]}) - E_{kr}^{[pp]} (E_{kk}^{[pp]} A + a_{rr} E_{rr}^{[pp]}) E_{rk}^{[pp]} - E_{rk}^{[pp]} A E_{kr}^{[pp]} + A E_{rr}^{[pp]} \quad (\text{by (5.17)}) \\
&= A - a_{rr} (E_{rr}^{[pp]} + E_{kk}^{[pp]}) - E_{rk}^{[pp]} A E_{kr}^{[pp]} + A E_{rr}^{[pp]}.
\end{aligned}$$

Therefore,

$$a_{rr} (E_{rr}^{[pp]} + E_{kk}^{[pp]}) + E_{rk}^{[pp]} A E_{kr}^{[pp]} = A E_{rr}^{[pp]}.$$

Comparing the (k, k) (resp. (r, r) , (k, r)) entry, we get $a_{rr} = 0$ (resp. $a_{kk} = 0$,

$a_{kr} = 0$). Since $r \in [m] - \{k\}$ is arbitrary, we have $f(\widetilde{[E_{kj}^{[p,p+1]}]})_{pp} = 0$.

We finish the proof for $\ell = 1$.

(2) Suppose (5.12) is true for all $\ell < k$. Now for any $(p, p+k) \in [I(\mathcal{L})]$, we have $[\widetilde{\mathcal{M}}_{p,p+1}, \widetilde{\mathcal{M}}_{p+1,p+k}] = \widetilde{\mathcal{M}}_{p,p+k}$ in $M_{\mathcal{L}}^0$, and by induction hypothesis,

$$\begin{aligned} f(\widetilde{A_{p,p+1}A_{p+1,p+k}}) &= f([\widetilde{A_{p,p+1}}, \widetilde{A_{p+1,p+k}}]) = [f(\widetilde{A_{p,p+1}}), \widetilde{A_{p+1,p+k}}] + [\widetilde{A_{p,p+1}}, f(\widetilde{A_{p+1,p+k}})] \\ &\in \widetilde{\mathcal{M}}_{p,p+k} + \sum_{i=1}^{p-1} \widetilde{\mathcal{M}}_{i,p+k} + \sum_{j=p+k+1}^t \widetilde{\mathcal{M}}_{pj}. \end{aligned}$$

Therefore, (5.12) is true for $\ell = k$.

(3) Overall, (5.12) is proved for all $(p, q) \in [I(\mathcal{L})]$ with $p < q$. \square

Lemma 5.12. *Suppose $\text{char}(\mathbb{F}) \neq 2, 3$. Let $f \in \text{Der}(M_{\mathcal{L}}^0)$. Then for any $1 \leq p < q \leq t$, there exists $X_{pq} \in \mathcal{M}_{pq}$ such that*

$$f(\widetilde{A_{ip}})_{iq} = -A_{ip}X_{pq}, \quad \text{for all } (i, p) \in [I(\mathcal{L})] \text{ and } \widetilde{A_{ip}} \in \widetilde{\mathcal{M}}_{ip} \cap M_{\mathcal{L}}^0, \quad (5.18)$$

$$f(\widetilde{A_{qj}})_{pj} = X_{pq}A_{qj}, \quad \text{for all } (q, j) \in [I(\mathcal{L})] \text{ and } \widetilde{A_{qj}} \in \widetilde{\mathcal{M}}_{qj} \cap M_{\mathcal{L}}^0. \quad (5.19)$$

The proof is similar to part (3) of the proof of Theorem 3.1 in Section 4.

Proof. Given $p < q$ in $[t]$, we consider the following four situations:

(1) Suppose $(q, j) = (t, t) \in [I(\mathcal{L})]$. For any $A_{tt}, B_{tt} \in \mathfrak{sl}_{tt}$,

$$f([\widetilde{A_{tt}}, \widetilde{B_{tt}}])_{pt} = [f(\widetilde{A_{tt}}), \widetilde{B_{tt}}]_{pt} + [\widetilde{A_{tt}}, f(\widetilde{B_{tt}})]_{pt} = f(\widetilde{A_{tt}})_{pt}B_{tt} - f(\widetilde{B_{tt}})_{pt}A_{tt}.$$

Applying Lemma 5.9 to the map $\phi : \mathfrak{sl}_{tt} \rightarrow \mathcal{M}_{pt}$ defined by $\phi(C) = f(\widetilde{C})_{pt}$, we can find $X_{pt} \in \mathcal{M}_{pt}$ such that $f(\widetilde{A_{tt}})_{pt} = X_{pt}A_{tt}$ for $A_{tt} \in \mathfrak{sl}_{tt}$.

(2) Similarly, when $(i, p) = (1, 1)$, there exists $Y_{1q} \in \mathcal{M}_{1q}$ such that $f(\widetilde{A_{11}})_{1q} = -A_{11}Y_{1q}$ for all $A_{11} \in \mathfrak{sl}_{11}$.

(3) Suppose $(q, j) \in [I(\mathcal{L})]$, $(q, j) \neq (t, t)$. Then $q < t$. Given any $j < j'$ in $[t]$, we have $(j, j'), (q, j'), (p, j), (p, j') \in [I(\mathcal{L})]$, and $\widetilde{\mathcal{M}}_{qj'} = \widetilde{\mathcal{M}}_{qj}\widetilde{\mathcal{M}}_{jj'} = [\widetilde{\mathcal{M}}_{qj}, \widetilde{\mathcal{M}}_{jj'}]$.

- If $q = j$, then for $A_{qj} \in \mathfrak{sl}_{qq}$ and $A_{jj'} \in \mathcal{M}_{jj'}$,

$$f(\widetilde{A_{qj}A_{jj'}})_{pj'} = f([\widetilde{A_{qj}}, \widetilde{A_{jj'}}])_{pj'} = [f(\widetilde{A_{qj}}), \widetilde{A_{jj'}}]_{pj'} + [\widetilde{A_{qj}}, f(\widetilde{A_{jj'}})]_{pj'} = f(\widetilde{A_{qj}})_{pj}A_{jj'}.$$

Applying Lemma 5.7 to the map $\phi : \mathcal{M}_{qj'} \rightarrow \mathcal{M}_{pj'}$ defined by $\phi(C) = f(\widetilde{C})_{pj'}$, and $\varphi : \mathfrak{sl}_{qq} \rightarrow \mathcal{M}_{pq}$ defined by $\varphi(D) = f(\widetilde{D})_{pj}$, there exists $X_{pq} \in \mathcal{M}_{pq}$ such that $f(\widetilde{A_{qj}})_{pj} = X_{pq}A_{qj}$ for $A_{qj} \in \mathfrak{sl}_{qq}$, and $f(\widetilde{A_{qj'}})_{pj'} = X_{pq}A_{qj'}$ for any $j' > j$ in $[t]$ and any $A_{qj'} \in \mathcal{M}_{qj'}$.

- If $q < j$, then for $A_{qj} \in \mathcal{M}_{qj}$ and $A_{jj'} \in \mathcal{M}_{jj'}$, we still have

$$f(\widetilde{A_{qj}A_{jj'}})_{pj'} = f([\widetilde{A_{qj}}, \widetilde{A_{jj'}}])_{pj'} = [f(\widetilde{A_{qj}}), \widetilde{A_{jj'}}]_{pj'} + [\widetilde{A_{qj}}, f(\widetilde{A_{jj'}})]_{pj'} = f(\widetilde{A_{qj}})_{pj}A_{jj'}.$$

Applying Lemma 4.2, there exists a (unique) $X_{pq} \in \mathcal{M}_{pq}$ such that $f(\widetilde{A_{qj}})_{pj} = X_{pq}A_{qj}$ for all $j > q$ in $[t]$.

(4) Suppose $(i, p) \in [I(\mathcal{L})]$ and $(i, p) \neq (1, 1)$. Similar to the proceeding argument, there exists $-Y_{pq} \in \mathcal{M}_{pq}$ such that $f(\widetilde{A_{ip}})_{iq} = -A_{ip}Y_{pq}$ for $(i, p) \in [I(\mathcal{L})]$ and $A_{ip} \in \mathcal{M}_{ip}$.

(5) For any $(i, p), (q, j) \in [I(\mathcal{L})]$, we have $[\widetilde{A_{ip}}, \widetilde{A_{qj}}] = 0$. So

$$\begin{aligned} 0 &= f([\widetilde{A_{ip}}, \widetilde{A_{qj}}])_{ij} = [f(\widetilde{A_{ip}}), \widetilde{A_{qj}}]_{ij} + [\widetilde{A_{ip}}, f(\widetilde{A_{qj}})]_{ij} \\ &= f(\widetilde{A_{ip}})_{iq}A_{qj} + A_{ip}f(\widetilde{A_{qj}})_{pj} = -A_{ip}Y_{pq}A_{qj} + A_{ip}X_{pq}A_{qj}. \end{aligned}$$

Therefore, $X_{pq} = Y_{pq}$. □

Now we are ready to prove Theorem 5.3.

Proof of Theorem 5.3. We have the Lie subalgebra decomposition

$$M_{\mathcal{L}} = \text{span}\{E_{11}^{[kk]} \mid (k, k) \in [I(\mathcal{L})]\} \ltimes M_{\mathcal{L}}^0.$$

Given $f \in \text{Der}(M_{\mathcal{L}}^0)$, we define $f^+(A) := f(A)$ for $A \in M_{\mathcal{L}}^0$. The next step is to define $f^+(\widetilde{E_{11}^{[kk]}})$ for each $(k, k) \in [I(\mathcal{L})]$ appropriately so that $f^+ \in \text{Der}(M_{\mathcal{L}})$. We will let

$$f^+(\widetilde{E_{11}^{[kk]}}) \in \widetilde{\mathfrak{sl}_{kk}} + \sum_{i=1}^{k-1} \widetilde{\mathcal{M}_{ik}} + \sum_{j=k+1}^t \widetilde{\mathcal{M}_{kj}}$$

and define the nonzero blocks of $f^+(\widetilde{E_{11}^{[kk]}})$ as follow.

- (1) The (k, k) block: it is easy to see that $f(\cdot)_{kk} : \mathfrak{sl}_{kk} \rightarrow \mathfrak{sl}_{kk}$, $A_{kk} \mapsto f(\widetilde{A_{kk}})_{kk}$, is a derivation of \mathfrak{sl}_{kk} . Since $\text{char}(\mathbb{F}) \neq 2$, there exists $X_{kk} \in \mathfrak{sl}_{kk}$ such that $f(\widetilde{A_{kk}})_{kk} = [X_{kk}, A_{kk}]$ for $A_{kk} \in \mathfrak{sl}_{kk}$. Define

$$f^+(\widetilde{E_{11}^{[kk]}})_{kk} := [X_{kk}, E_{11}^{[kk]}]. \quad (5.20)$$

- (2) The (i, k) block, $i < k$: by Lemma 5.12, there exists $X_{ik} \in \mathcal{M}_{ik}$ such that $f(\widetilde{A_{kj}})_{ij} = X_{ik}A_{kj}$ for any $(k, j) \in [I(\mathcal{L})]$. Define

$$f^+(\widetilde{E_{11}^{[kk]}})_{ik} := X_{ik}E_{11}^{[kk]} \quad \text{for all } i \in [k-1]. \quad (5.21)$$

- (3) The (k, j) block, $k < j$: by Lemma 5.12, there exists $X_{kj} \in \mathcal{M}_{kj}$ such that for all $(i, k) \in [I(\mathcal{L})]$ we have $f(\widetilde{A_{ik}})_{ij} = -A_{ik}X_{kj}$. Define

$$f^+(\widetilde{E_{11}^{[kk]}})_{kj} := -E_{11}^{[kk]}X_{kj} \quad \text{for all } k < j \leq t. \quad (5.22)$$

The above process uniquely defines a linear map $f^+ \in \text{End}(M_{\mathcal{L}})$ such that $f^+|_{M_{\mathcal{L}}^0} = f$. Next we verify that $f^+ \in \text{Der}(M_{\mathcal{L}})$. It suffices to prove that for every $(i, j) \in [I(\mathcal{L})]$,

$$f^+(\widetilde{[E_{11}^{[kk]}, \widetilde{A_{ij}}]}) = [f^+(\widetilde{E_{11}^{[kk]}}), \widetilde{A_{ij}}] + \widetilde{[E_{11}^{[kk]}, f^+(\widetilde{A_{ij}})]} \quad \text{for all } \widetilde{A_{ij}} \in \widetilde{\mathcal{M}_{ij}} \cap M_{\mathcal{L}}^0. \quad (5.23)$$

Denote

$$X_k := \widetilde{X_{kk}} + \sum_{i=1}^{k-1} \widetilde{X_{ik}} + \sum_{j=k+1}^t \widetilde{X_{kj}}. \quad (5.24)$$

Then (5.20), (5.21), and (5.22) imply that $f^+(\widetilde{E_{11}^{[kk]}}) = [X_k, \widetilde{E_{11}^{[kk]}}]$. So (5.23) is equivalent to

$$f(\widetilde{[E_{11}^{[kk]}, \widetilde{A_{ij}}]}) = [[X_k, \widetilde{E_{11}^{[kk]}}], \widetilde{A_{ij}}] + \widetilde{[E_{11}^{[kk]}, f(\widetilde{A_{ij}})]} \quad \text{for all } \widetilde{A_{ij}} \in \widetilde{\mathcal{M}_{ij}} \cap M_{\mathcal{L}}^0. \quad (5.25)$$

We will prove (5.25) for each block $(i, j) \in [I(\mathcal{L})]$:

- (1) $(k, k) \in [I(\mathcal{L})]$: the matrices X_{kk} , X_{ik} ($i < k$), and X_{kj} ($k < j$) satisfy that

$$f(\widetilde{A_{kk}}) = [X_k, \widetilde{A_{kk}}] \quad \text{for all } A_{kk} \in \mathfrak{sl}_{kk},$$

where X_k is given by (5.24). Therefore, (5.25) is true for $(i, j) = (k, k) \in [I(\mathcal{L})]$.

(2) (k, j) , $k < j \leq t$: when $(i, j) = (k, j)$, we have

$$[\widetilde{E_{11}^{[kk]}}, \widetilde{A_{kj}}] = \widetilde{E_{11}^{[kk]} A_{kj}} = \widetilde{E_{12}^{[kk]} E_{21}^{[kk]} A_{kj}} = [\widetilde{E_{12}^{[kk]}}, [\widetilde{E_{21}^{[kk]}}, \widetilde{A_{kj}}]].$$

So (5.25) is equivalent to the following equalities:

$$\begin{aligned} & f([\widetilde{E_{12}^{[kk]}}, [\widetilde{E_{21}^{[kk]}}, \widetilde{A_{kj}}]]) = [[X_k, \widetilde{E_{11}^{[kk]}}, \widetilde{A_{kj}}] + [\widetilde{E_{11}^{[kk]}}, f(\widetilde{A_{kj}})] \\ \iff & f(\widetilde{E_{12}^{[kk]}}) \widetilde{E_{21}^{[kk]} A_{kj}} + \widetilde{E_{12}^{[kk]} f(E_{21}^{[kk]}) A_{kj}} + \widetilde{E_{12}^{[kk]} E_{21}^{[kk]} f(A_{kj})} = [X_k, \widetilde{E_{11}^{[kk]}}] \widetilde{A_{kj}} + \widetilde{E_{11}^{[kk]} f(A_{kj})} \\ \iff & f(\widetilde{E_{12}^{[kk]}}) \widetilde{E_{21}^{[kk]} A_{kj}} + \widetilde{E_{12}^{[kk]} f(E_{21}^{[kk]}) A_{kj}} = [X_k, \widetilde{E_{11}^{[kk]}}] \widetilde{A_{kj}} \quad (\text{for all } A_{kj} \in \mathcal{M}_{kj}) \\ \iff & f(\widetilde{E_{12}^{[kk]}}) \widetilde{E_{21}^{[kk]}} + \widetilde{E_{12}^{[kk]} f(E_{21}^{[kk]})} = [X_k, \widetilde{E_{11}^{[kk]}}] \\ \iff & [X_k, \widetilde{E_{12}^{[kk]}}] \widetilde{E_{21}^{[kk]}} + \widetilde{E_{12}^{[kk]} [X_k, E_{21}^{[kk]}]} = [X_k, \widetilde{E_{11}^{[kk]}}]. \end{aligned}$$

The last equality is obviously true.

(3) (i, k) , $1 \leq i < k$: similarly, we can prove (5.25) for the case $(i, j) = (i, k)$.

(4) $(i, j) \in [I(\mathcal{L})]$, $i \neq k$, $j \neq k$: the left side of (5.25) is zero. We investigate the right side of (5.25) in three cases:

(a) $i \leq j < k$: the only possibly nonzero block in the right side of (5.25) is the (i, k) block, which is

$$\begin{aligned} [[X_k, \widetilde{E_{11}^{[kk]}}, \widetilde{A_{ij}}]_{ik} + [\widetilde{E_{11}^{[kk]}}, f(\widetilde{A_{ij}})]_{ik} &= -A_{ij}[X_k, \widetilde{E_{11}^{[kk]}}]_{jk} - f(\widetilde{A_{ij}})_{ik} E_{11}^{[kk]} \\ &= -A_{ij}[X_k, \widetilde{E_{11}^{[kk]}}]_{jk} + A_{ij} X_{jk} E_{11}^{[kk]} \quad (\text{by Lemma 5.12}) \\ &= -A_{ij} X_{jk} E_{11}^{[kk]} + A_{ij} X_{jk} E_{11}^{[kk]} \quad (\text{by (5.24)}) \\ &= 0. \end{aligned}$$

So (5.25) is done for this case.

(b) $k < i \leq j$: similarly, we can prove (5.25) for this case.

(c) $i < k < j$: the right side of (5.25) is

$$[[X_k, \widetilde{E_{11}^{[kk]}}, \widetilde{A_{ij}}] + [\widetilde{E_{11}^{[kk]}}, f(\widetilde{A_{ij}})] = 0 + 0 = 0.$$

So (5.25) holds.

Overall, we have proved (5.25). Therefore, $f^+ \in \text{Der}(M_{\mathcal{L}})$ and $f^+|_{M_{\mathcal{L}}^0} = f$. By Theorem 3.1, there is $X \in M_{\mathcal{L}_B}$ such that $f(B) = [X, B]$ for all $B \in M_{\mathcal{L}}^0$. \square

REFERENCES

- [1] D. Brice, *On derivations of parabolic Lie algebras*, submitted, [arXiv:1504.08286 \[math.RA\]](#).
- [2] D. Brice, H. Huang (2014), *On zero product determined algebra*, Linear and Multilinear Algebra, **63** (2015) 326–342 DOI:10.1080/03081087.2013.866668.
- [3] Wai-Shun Cheung (2003) Lie Derivations of Triangular Algebras, Linear and Multilinear Algebra, **51:3**, 299–310, DOI: 10.1080/0308108031000096993
- [4] J. Dixmier, W.G. Lister, *Derivations of Nilpotent Lie algebras*, Proceedings of the American Mathematical Society, Vol. **8**, No **1** (1957) 155–158.
- [5] Y. Du, Y. Wang *Lie derivations of generalized matrix algebras*, Linear Algebra and Its Applications, Vol. **437** (2012) 2719–2726.
- [6] N. Jacobson, *A Note on automorphisms and derivations of Lie algebras*, Proceedings of the American Mathematical Society, Vol. **6** (1955) 281–283.

- [7] S. Ou, D. Wang, R. Yao, *Derivations of the Lie algebra of strictly upper triangular matrices over a commutative ring*, Linear Algebra and Its Applications **424** (2007) 378–383.
- [8] D. Wang, Q. Yu, *Derivation of the parabolic subalgebras of the general linear Lie algebra over commutative ring*, Linear Algebra and Its Applications **418** (2006) 763–774.
- [9] Z. X. Chen, *Generalized derivations on parabolic subalgebras of general linear Lie algebras*, Acta Math. Sci. Ser. B Engl. Ed. **34** (2014), no. 3, 814–828.
- [10] D. Wang, S. Ou, Q. Yu, *Derivations of the intermediate Lie algebras between the Lie algebra of diagonal matrices and that of upper triangular matrices over a commutative ring*, Linear Multilinear Algebra **54** (2006), no. 5, 369–377.
- [11] J. Li, Y. Cao, Z. Li, *Triple and generalized triple derivations of the parabolic subalgebras of the general linear Lie algebra over a commutative ring*, Linear Multilinear Algebra **61** (2013), no. 3, 337–353.
- [12] P. S. Ji, X. L. Yang, J. H. Chen, *Biderivations of the algebra of strictly upper triangular matrices over a commutative ring*, J. Math. Res. Exposition **31** (2011), no. 6, 965–976.
- [13] X. Y. Kong, L. L. Zhou, N. N. Li, *Lie triple derivations of general linear Lie algebras over a commutative ring*, J. Math. (Wuhan) **32** (2012), no. 4, 663–668.
- [14] D. Benkovič, *Lie triple derivations on triangular matrices*, Algebra Colloq. **18** (2011), Special Issue No.1, 819–826.

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